Determinants - Fundamental Rules[1]
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Why to look into determinants:

1. det $A \neq 0$ then A is invertible
2. det $A =$ box volume in n dimensional space where edges of the box are either all rows or all columns of A
3. det $A =$ product of pivots. Compute $(i+1)^{th}$ pivot by evaluating det $A_{i \times i}$ and det $A_{(i+1)\times(i+1)}$ extracted from A
4. A^{-1} shows dependence of x on particular observation in tuple b

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Practical examples of application of determinants in engineering is the test for element quality in FEM using Jacobian, test for topology in sweeped geometries, volume estimations for parallelopiped, volume change in case of deformation problems, etc.

Fundamental 3 Properties of Determinant:

1. det $I = 1$
2. sign of det A flips if any two rows of A are interchanged
3. Determinant is linear in each row separately

1. det $I = 1$
2. $$det \begin{bmatrix} ----A_1--- \\ \cdots \\ ----A_k--- \\ \cdots \\ ---A_p--- \\ \cdots \\ ---A_m--- \end{bmatrix} = -det \begin{bmatrix} ----A_1--- \\ \cdots \\ ----A_p--- \\ \cdots \\ ---A_k--- \\ \cdots \\ ---A_m--- \end{bmatrix}$$
3. $$det \begin{bmatrix} a+a' & b+b' \\ c & d \end{bmatrix} = det \begin{bmatrix} a & b \\ c & d \end{bmatrix} + det \begin{bmatrix} a' & b' \\ c & d \end{bmatrix}$$

[1] Reference: Gilbert Strang, Linear Algebra and Its Applications, 4th Edition.

$$det \begin{bmatrix} ta & tb \\ c & d \end{bmatrix} = t * det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

4. Two equal rows in $A \Longrightarrow detA = 0$ Hint: use property 2. Before interchanging and after interchanging the identical rows what should happen to the determinant?

5. No change in determinant due to row operations

$$det \begin{bmatrix} a - \alpha c & b - \alpha d \\ c & d \end{bmatrix} = det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

   Hint: Use property 3 and 4

6. zero row $\Longrightarrow detA = 0$ Hint:What happens if one adds to non-zero row to zero row and uses Rule4 and Rule5

7. For a triangular matrix A , $det\ A$ =multiplication of all diagonal elements. Hint: Row operations can reduce it to diagonal matrix D (Rule 5). Then use Rule 3 to reduce D to identity matrix I for which Rule1 is defined.

8. singular A means $\Longrightarrow detA = 0$ and invertible A $\Longrightarrow detA \neq 0$ Hint: Use the row operations to either make some row a zero row or reduce it to triangular matrix

9. $det\,[AB] = detA * detB$ Hint: for diagonal matrix D, $detDB = detD * detB$ (Rule3) and we know that we can reduce A to diagonal D using row operations where determinants remains the same.

10. $detA^T = detA$ Hint: use PA=LDU decomposition followed by using Rule 9 $detP * detA = detL * detD * detU$ as well as $detA^T * detP^T = detU^T * detD^T * detL^T$. Note that L, U are triangular with all diagonal elements being ones. P is permutation matrix. Hence, $detP = \pm 1$ and $PP^T = I$