IIT DHARWAD AUTUMN 2019

Engg. Maths for Advanced Studies.

Module - Vector Calculus

Sample Solutions - Final Exam

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VecCal
Au19

Q.1

When maxima or minima occurs at a point, the function value becomes stationary in the neighborhood of that point. i.e. gradient vector becomes a null vector.

$\therefore \quad \nabla f = \vec{0}$ for maxima / minima to occur.

$$= 0\hat{i} + 0\hat{j}$$

$$\therefore \quad \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} = 0\hat{i} + 0\hat{j}$$

$$\therefore \quad \frac{\partial f}{\partial x} = 0 \quad \& \quad \frac{\partial f}{\partial y} = 0$$

Given, $f(x,y) = (x-2)^4 + (x-2y)^2$

$$\frac{\partial f}{\partial x} = 4(x-2)^3 + 2(x-2y)$$

$$\frac{\partial f}{\partial y} = 2(x-2y).(-2) = -4(x-2y)$$

Hence, $\frac{\partial f}{\partial y} = 0 \Rightarrow x = 2y$

substitute in $\frac{\partial f}{\partial x} = 0 \Rightarrow 4(2y-2)^3 + 2(2y-2y) = 0$

$$\therefore \quad y = 1 \Rightarrow x = 2.$$

$\therefore$ Maxima / Minima will occur at point $(2,1)$.

(Note → Observation of $f(x,y)$, which contains both positive terms, one of which is 4^{th} power & other is 2^{nd} power suggest that this is minima.) &

<u>This</u> should be <u>THE</u> ONLY MINIMA.

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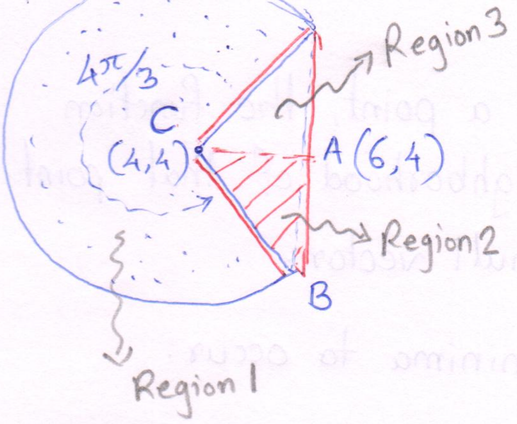
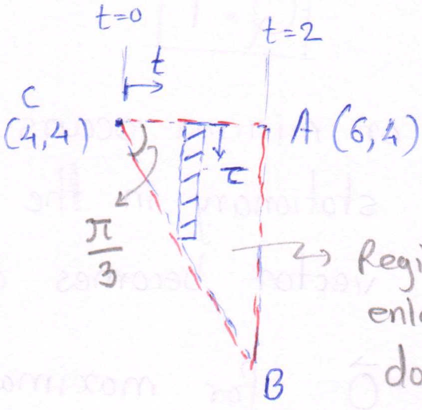
Q.2

There are multiple ways in ~~th~~ which this question can be answered. It depends on the symmetry used & selection of area for double integral. & area ~~used~~ selected to apply standard formula.





Region 2 enlarged to show double integral approach.

$$\int_0^2 \int_0^{t \tan(\frac{\pi}{3})} d\tau \cdot dt$$

Split given area in two equal halfves about verticle dividing line passing through A (6,4) & B points. This area is shown above on left. This area is further split into region 1, region 2 & region 3. Note that due to symmetry about horizontal line passing through point A ≥ i.e. AC line, region 2 & region 3 have equal area.

Area of region 1 = $\frac{2}{3}$ of area of entire circle of radius (R=4) ∴ (angle = $\frac{4\pi}{3}$).

$$= \frac{2}{3} \pi R^2$$

$$A(\text{region } 1) = \frac{32}{3}\pi \quad \therefore (R=4)$$

$\boxed{Q.2}$

Area of region 2 (= Area of region 3)

$$= \int_4^6 \left[\int_0^{t \cdot \tan(\frac{\pi}{3})} d\tau \right] dx$$

(Note -: τ variable is dummy variable for int. in vertical direction & $'t'$ is dummy variable as shown in figure).

$$= \int_0^2 \int_0^{\sqrt{3}t} d\tau \, dt$$

$$= \int_0^2 \sqrt{3} t \, dt = \frac{\sqrt{3}}{2} \left[t^2 \right]_0^2 = 2\sqrt{3}$$

Hence,

Area of (region 1 + region 2 + region 3) $= \frac{32}{3}\pi + 2\sqrt{3} + 2\sqrt{3} = \frac{32}{3}\pi + 4\sqrt{3}$

$\therefore$ Considering symmetry, $\boxed{\text{Total Area} = \frac{64}{3}\pi + 8\sqrt{3}}$

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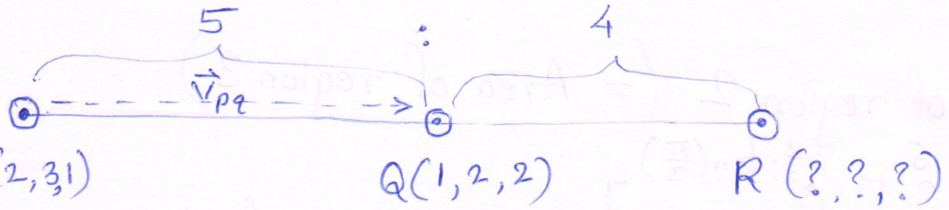
Q.3

(a)



$$\vec{p} = 2\hat{i} + 3\hat{j} + \hat{k}$$

$$\vec{q} = \hat{i} + 2\hat{j} + 2\hat{k}$$

$$\therefore \vec{V}_{pq} = \vec{q} - \vec{p} = -\hat{i} - \hat{j} + \hat{k}$$

Aside,
$$\Rightarrow \|\vec{V}_{pq}\| = \sqrt{1+1+1} = \sqrt{3}$$
i.e. $\|PQ\| = \sqrt{3}$.

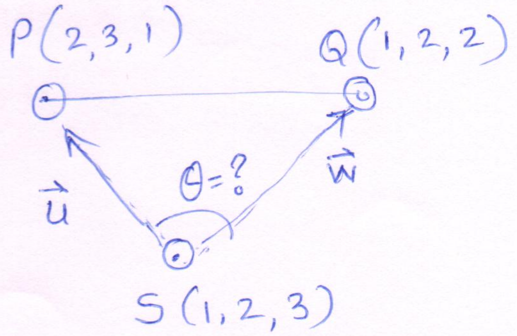
Position vector $\vec{r}$ for point R:

$$\therefore \vec{r} = \vec{p} + \frac{9}{5}\vec{V}_{pq} = 2\hat{i} + 3\hat{j} + \hat{k} + \frac{9}{5}(-\hat{i} - \hat{j} + \hat{k})$$

$$\vec{r} = \frac{1}{5}\hat{i} + \frac{6}{5}\hat{j} + \frac{14}{5}\hat{k}$$

$$\therefore R \Rightarrow R\left(\frac{1}{5}, \frac{6}{5}, \frac{14}{5}\right)$$

(b)



$$\vec{s} = \hat{i} + 2\hat{j} + 3\hat{k}$$

$$\vec{u} = \vec{p} - \vec{s} = \hat{i} + \hat{j} - 2\hat{k}$$

$$\vec{w} = \vec{q} - \vec{s} = -\hat{k}$$

$$\therefore \vec{u} \cdot \vec{w} = 2.$$

$$\|\vec{u}\| = \sqrt{6} \quad \& \quad \|\vec{w}\| = 1$$

$$\therefore \cos\theta = \frac{\vec{u} \cdot \vec{w}}{\|\vec{u}\| \cdot \|\vec{w}\|} = \frac{2}{\sqrt{6}} = \sqrt{\frac{2}{3}}$$

$$\therefore \boxed{\theta = \cos^{-1}\left(\sqrt{\frac{2}{3}}\right)}$$

$\boxed{Q.4}$

Note formula # 5 provided

$$\int_C \vec{f} \cdot d\vec{r} = \int_C \left[P\,dx + Q\,dy \right] \quad \left(\vec{f}(x,y) = P(x,y)\,\hat{i} + Q(x,y)\,\hat{j} \right)$$

Here, $\vec{f}(x,y) = -2y\,\hat{i} + x\hat{j} \implies P(x,y) = -2y$
$Q(x,y) = x$

$$\therefore \int_C \vec{f} \cdot d\vec{r} = \int_C \left[P\,dx + Q\,dy \right] = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

$$= \iint_R \left(1 - (-2) \right) dx\,dy = 3 \times \iint_R dx\,dy$$

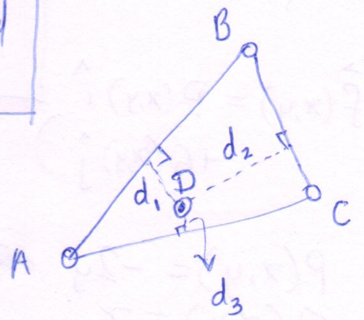
= 3 × area enclosed by the closed curve.

Area of given shape was calculated in Q.2 as

$$\text{Area} = \frac{64}{3}\pi + 8\sqrt{3}.$$

Hence $\boxed{\oint_C \vec{F} \cdot d\vec{r} = 64\pi + 24\sqrt{3}}$

Q.5



Equivalent simplified abstraction of the problem:

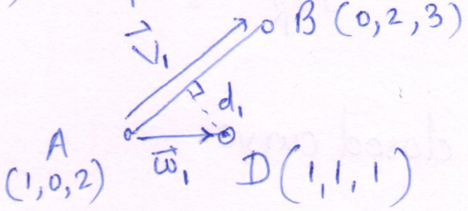
Given,

$A(1,0,2)$, $B(0,2,3)$, $C(3,3,0)$, $D(1,1,1)$

find maximum of $\{d_1, d_2, d_3\}$

(point #2 on first page of question paper provides the required formula readily to get distance of a point from line.)

i) Computation of d_1



$$\vec{a} = \hat{i} + 0\hat{j} + 2\hat{k}$$

$$\vec{b} = 0\hat{i} + 2\hat{j} + 3\hat{k}$$

$$\vec{d} = \hat{i} + \hat{j} + \hat{k}$$

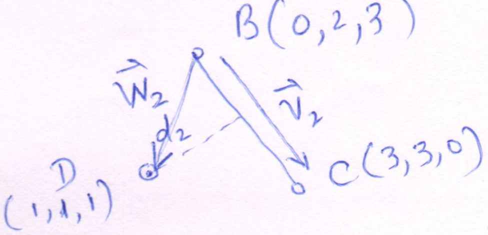
$$\therefore \vec{v}_1 = \vec{b} - \vec{a} = -\hat{i} + 2\hat{j} + \hat{k} \Rightarrow \|\vec{v}_1\| = \sqrt{6}$$

$$\vec{w}_1 = \vec{d} - \vec{a} = 0\hat{i} + \hat{j} - \hat{k}$$

$$\therefore \vec{v}_1 \times \vec{w}_1 = \begin{vmatrix} i & j & k \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{vmatrix} = -3\hat{i} - \hat{j} - \hat{k} \Rightarrow \|\vec{v}_1 \times \vec{w}_1\| = \sqrt{11}$$

Hence, $$d_1 = \frac{\|\vec{v}_1 \times \vec{w}_1\|}{\|V_1\|} = \frac{\sqrt{11}}{\sqrt{6}} = \sqrt{\frac{11}{6}} \quad \text{——— (5a)}$$

ii) Computation of d_2,



$$\vec{b} = 0\hat{i} + 2\hat{j} + 3\hat{k}$$

$$\vec{c} = 3\hat{i} + 3\hat{j} + 0\hat{k}$$

$$\vec{d} = \hat{i} + \hat{j} + \hat{k}$$

$$\vec{V}_2 = \vec{c} - \vec{b} = 3\hat{i} + \hat{j} - 3\hat{k} \Rightarrow \|\vec{v}_1\| = \sqrt{19}$$

$$\vec{W}_2 = \vec{d} - \vec{b} = \hat{i} - \hat{j} - 2\hat{k}$$

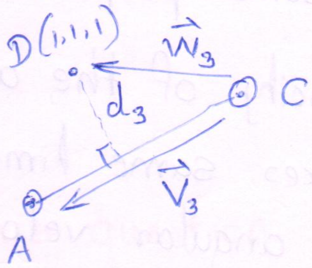
$\boxed{Q.5}$ continued

$$\vec{v}_2 \times \vec{w}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -3 \\ 1 & -1 & -2 \end{vmatrix} = -5\hat{i} + 3\hat{j} - 4\hat{k}$$

$$\Rightarrow \|\vec{v}_2 \times \vec{w}_2\| = \sqrt{50}$$

$$d_2 = \frac{\|\vec{v}_2 \times \vec{w}_2\|}{\|\vec{v}_2\|} = \frac{\sqrt{50}}{\sqrt{19}} = \sqrt{\frac{50}{19}} \quad \text{—— } 5b$$

iii) Computation of d_3



$$\vec{c} = 3\hat{i} + 3\hat{j} + 0\hat{k}$$
$$\vec{a} = \hat{i} + 0\hat{j} + 2\hat{k}$$
$$\vec{d} = \hat{i} + \hat{j} + \hat{k}$$

$$\vec{v}_3 = \vec{a} - \vec{c} = -2\hat{i} - 3\hat{j} + 2\hat{k} \Rightarrow \|\vec{v}_3\| = \sqrt{17}$$

$$\vec{w}_3 = \vec{d} - \vec{c} = -2\hat{i} - 2\hat{j} + \hat{k}$$

$$\therefore \vec{v}_3 \times \vec{w}_3 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & -3 & 2 \\ -2 & -2 & 1 \end{vmatrix} = \hat{i} - 2\hat{j} - 2\hat{k}$$

$$\therefore \|\vec{v}_3 \times \vec{w}_3\| = \sqrt{9}$$

$$\therefore d_3 = \frac{\|\vec{v}_3 \times \vec{w}_3\|}{\|\vec{v}_3\|} = \sqrt{\frac{9}{17}} \quad \text{—— } 5c$$

Considering d_1, d_2 & d_3 in 5a, 5b & 5c respectively

$$\text{Maximum } \{d_1, d_2, d_3\} = d_2 = \sqrt{\frac{50}{19}}.$$

Hence, $\boxed{\text{minimum necessary range} = \sqrt{\frac{50}{19}}}$

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$\boxed{Q.9}$

TRUE - it is simply connected.

$\boxed{Q.10}$

(a) $\vec{v} = y\hat{i} - x\hat{j}$

$$\vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = 0\hat{i} + 0\hat{j} \cancel{+} -2\hat{k} \neq \vec{0}$$

Hence, flow field is NOT irrotational

(b)
$$\nabla \cdot \vec{v} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(-x) = 0.$$

Hence, flow is incompressible

$\boxed{Q.12}$

(a) P → Osculating plane

(b) c vector → tangent vector

(c) a vector → principal normal vector

(d) R → normal plane