IIT DHARWAD
Autumn 2019
Engg. Math. for Advanced Studies

ODE - Quiz - Solutions

(I)

Q.1

Taylor Series :

Ans: a) $$y(x_0+\Delta x) = \sum_{n=0}^{\infty} \frac{(\Delta x)^n}{n!}\left(y^{(n)}(x_0)\right)$$

Q.2

This is a first order ODE typically given as

$$y' = ay + q(t)$$

(more ~~generally~~ generically, $y' = a(t)\cdot y + q(t)$)

with $a = \frac{6}{100}$ $t_0 = 0$, $t_1 = 2$, $t_2 = 5$

$$q(t) = \delta(t) \times 10000 + \delta(t-2) \times 5000$$ → key step.

Hence, diff. eqn.

(a) $$\boxed{y' = 0.06y + 10000\,\delta(t) + 5000\,\delta(t-2)}$$

(b) Generic solution for $y' - ay = \delta(t) \Rightarrow y(t) = e^{at}$

$y' - ay = \delta(t-T) \Rightarrow$ delayed dirac-delta at time $t=T$ instead of $t=0$.

$$y = 10000\,e^{0.06t} \quad \text{.. for } t \le 2$$

$$y = 10000\,e^{0.06t} + 5000\,e^{0.06(t-2)}$$

(for $t > 2$) (using principle of superimposition as this is Linear ODE).

$$\therefore y_5 = 10000\,e^{0.06\times 5} + 5000\,e^{0.06\times 3}$$
$$= 13498 + 5986$$
$$\boxed{y_5 = 19484 \text{ Rs.}}$$

P02
ODE-Qz-Soln.

Q.3

(a) $y' = 2-4y$

$q(t) = q$ (i.e. const.)
$= 2$

$a = -4$ (i.e. const.) & $y(0) = 3$

For $y' = ay + q(t)$ solution: $y = y(0)e^{at} + e^{at}\int_0^t e^{-as} q(s)\,ds.$

$= 3e^{-4t} + 2e^{-4t}\int_0^t e^{-as}\,ds$ ... $(q = 2 = \text{const.})$

$\therefore \quad y = 3e^{-4t} + 2e^{-4t}\left[\frac{e^{-as}}{-a}\right]_0^t$

$= 3e^{-4t} + \frac{2e^{-4t}}{+4}\left[e^{+4t} - e^{0}\right]$

$$\boxed{y = 3e^{-4t} + \frac{1}{2}e^{-4t}\left[-1 + e^{+4t}\right]}$$

(b) $y = \underbrace{\cancel{3e^{-4t}}}_{=0 \text{ at } t\to\infty} + \frac{1}{2} - \underbrace{\cancel{\frac{1}{2}e^{-4t}}}_{=0 \text{ at } t\to\infty}$

$\therefore \quad y_\infty = \frac{1}{2}$

Q.4

(a) $y' - 8y = 2H(t-3)$

Null sol^n for $y' - 8y = 0 \Rightarrow y = k\cdot e^{8t}$

Particular sol^n for $y' - 8y = 2H(t-3)$

$y_p(t) = e^{at}\int_0^t e^{-as}\cdot(2H(t-3))\,ds = e^{at}\left[\int_0^3 e^{-as}\cdot 0\cdot ds + \int_3^t e^{-as}\cdot 2\,ds\right]$

$= e^{at}\left[0 + 2\int_3^t e^{-as}\,ds\right] = 2e^{at}\left[\frac{e^{-as}}{-a}\right]_3^t = \frac{2e^{8t}}{-8}\left[e^{-8s}\right]_3^t$

(a) continued...

$$y_p(t) = \frac{-e^{8t}}{4}\left[e^{-8t} - e^{-24}\right]$$

$$= \frac{e^{8t}}{4}\left[e^{-24} - e^{-8t}\right]$$

$$\therefore\ y = y_n + y_p = ke^{8t} + \frac{e^{8t}}{4}\left[e^{-24} - e^{-8t}\right]$$

$$\boxed{y = ke^{8t} + \frac{1}{4}\left[e^{8t-24} - 1\right]}$$

k to be determined based on initial conditions / boundary

(b) $y' - 8y = \cancel{2}\, e^{2t}$

Null solution for $y' - 8y = 0 \Rightarrow y = ke^{8t}$

Particular solution to $y' - 8y = e^{2t}$:

$$y_p = e^{at}\int_0^t e^{-as}\cdot q(s)\, ds$$

$$= e^{8t}\int_0^t e^{-8s}\cdot e^{2s}\, ds$$

$$= e^{8t}\int_0^t e^{-6s}\, ds = e^{8t}\left[\frac{e^{-6s}}{-6}\right]_0^t = e^{8t}\cdot\frac{1}{6}\left(1 - e^{-6t}\right)$$

$$= \frac{1}{6}\left(e^{8t} - e^{14t}\right)$$

$$\therefore\ y = ke^{8t} + \frac{1}{6}\left(e^{8t} - e^{\cancel{14}\,2t}\right) = \left(k + \frac{1}{6}\right)e^{8t} - \frac{e^{2t}}{6}$$

$$\boxed{y = \left(k + \frac{1}{6}\right)e^{8t} - \frac{e^{2t}}{6}}$$

P.04
ODE-Qz-Soln.

Q.5

$$y' - 5y = \cos(4t) + 2\sin(4t)$$

if $y = M\cos(\omega t) + N\sin(\omega t) \Rightarrow y' = -M\omega\sin(\omega t) + N\omega\cos(\omega t)$.

$\therefore y' - 5y = (-M\omega - 5N)\sin(\omega t) + (N\omega - 5M)\cos(\omega t)$.

Comparing sin & cos terms,

$N\omega - 5M = 1$

$-M\omega - 5N = 1$

$\therefore \omega(M+N) + 5(N-M) = 0$.

$\therefore \omega(M+N) = 5(M-N) \Rightarrow M+N = \frac{5}{4}(M-N)$ for $\omega = 4$ in given problem.

$\therefore \frac{9}{4}N = \frac{1}{4}M \Rightarrow M = 9N$.

$\therefore N\omega - 5M = 1$ becomes (with $\omega = 4$ & $M = 9N$)

$4N - 45N = 1$

$\therefore N = -\frac{1}{41}$ & $M = -\frac{9}{41}$

$$\therefore \boxed{y = -\frac{9}{41}\cos(4t) - \frac{1}{41}\sin(4t)}$$

Q.6

$$\cos(2t) + \sin(2t) = R\cos(2t - \phi).$$

Compare LHS with $A\cos(2\omega t) + B\sin(\omega t)$:

$A = 1 \quad B = 1 \quad \omega = 2$.

$R = \sqrt{A^2+B^2} = \sqrt{2} \qquad \phi = \tan^{-1}\left(\frac{B}{A}\right) = \frac{\pi}{4}$

$$\therefore \boxed{\cos(2t) + \sin(2t) = \sqrt{2}\cos\left(2t - \frac{\pi}{4}\right)}$$

Q. ~~6~~7

What is α & G in $Ge^{-i\alpha} = \frac{1}{re^{i\alpha}}$ for $re^{i\alpha} = \sqrt{3}i + 1$

Task 1 is to convert $1 + \sqrt{3}i$ into polar for $re^{i\alpha}$.

$$r = \sqrt{1+3} = 2 \quad \& \quad \alpha = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$$

$$\therefore re^{i\alpha} = 2e^{i\left(\frac{\pi}{3}\right)} = 2\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]$$

$$= 2\left[\frac{1}{2} + \frac{\sqrt{3}i}{2}\right] = \sqrt{3}i + 1$$

... (quick verification)

$$\therefore \boxed{G = \frac{1}{r} = \frac{1}{2} \quad \alpha = \frac{\pi}{3}}$$

Q. 8

a) $(y'')^2 + (y')^3 + 3y - \cos(x) = 0$

Order → 2

Degree → 2

Non-Homogeneous

Non-linear

b) $(y'')^2 - 3y'''y^3 = 0$

Order → 3

Degree → ~~2~~ 1

Homogeneous

Non-Linear.

P06
QDE-Qz-Soln.
Autumn 2019

$\boxed{Q.9}$

The key understanding for this question is :

i) Given plot has y along horizontal axis & $\frac{dy}{dt}$ along vertical axis.

ii) $\frac{dy}{dt}$ i.e. rate of change in y is only function of y itself.

if $f(y) > 0 \Rightarrow$ y will increase $\Rightarrow$ state ~~we~~ will shift towards right on horizontal axis

if $f(y) < 0 \Rightarrow$ y will decrease.

if $f(y) = 0 \Rightarrow$ y will stay the same.

It is equillibrium point however it could be ~~both~~ either stable or unstable equillibrium. Small perturbation will affect unstable equillibrium. e.g. at $y=2$, $y'=0 \Rightarrow$ a small positive perturbation will have $y'>0$ & y will start shifting to right. till $y=0$ when $y'=0$. Similarly for negative perturbation at $y=2$.

a) $y(0) = 1.01 \Rightarrow y' < 0 \Rightarrow$ y will decrease.

$\Rightarrow$ when $y = -\varepsilon$ i.e. just below zero, $y' > 0$ & hence y will increase.

$\Rightarrow y_\infty = 0$.

Similarly

b) $y(0) = 2.01 \Rightarrow y_\infty = 6$

c) $y(0) = 5.01 \Rightarrow y_\infty = 6$

Note $\rightarrow$ Equillibrium is stable when ~~$y'' < 0$~~ $\frac{d}{dy}(y') < 0$ i.e. $\frac{d}{dy}(f(y)) < 0$ i.e. ~~$f'(y) < 0$~~.

$\boxed{Q.10}$

Exactness condition for $M\,dx + N\,dy = 0$

i.e. $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ . . (Note $M = M(x,y)$, $N = N(x,y)$).

Given eqn. $(1-2xy^2)\,dx - (2x^2y)\,dy = 0$.

$\therefore M = 1-2xy^2$

$N = -2x^2y$

$\therefore \dfrac{\partial M}{\partial y} = -4xy \qquad \dfrac{\partial N}{\partial x} = -4xy$

$\therefore \boxed{\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} \Rightarrow \text{Exactness condition is satisfied}}$

Aside

if ~~we~~ there exists u such that $M = \dfrac{\partial u}{\partial x}$ & $N = \dfrac{\partial u}{\partial y}$

$M\,dx + N\,dy = 0 \Rightarrow \dfrac{\partial u}{\partial x}dx + \dfrac{\partial u}{\partial y}\cdot dy = 0 \Rightarrow \cancel{du}$

$\therefore du = 0$

↳ this is total differential of $u(x,y)$.

$\Rightarrow u(x,y) = \text{constant}$. will be a solution to $M\,dx + N\,dy = 0$.

$\dfrac{\partial M}{\partial y} = \dfrac{\partial^2 u}{\partial x \partial y}$

$\dfrac{\partial N}{\partial x} = \dfrac{\partial^2 u}{\partial x \partial y}$

if $u(x,y)$ exists, $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ i.e. the exactness condition

IIT Dharwad

Engineering Mathematics for Advanced Studies

Autumn 2019
Date – 16th Sept. 2019
Marks – 20

ODE Quiz01
Time – 25 minutes

Rule for absentee - Minimum 30% penalty, discuss reasons absence in person to get a chance for re-test.

Note –

1. Considering the time allotted for the quiz, to be on the safer side, student may want to first go through all questions and identify the ones that would not take much time to answer. Answer these first and then work on the relatively tedious questions.
2. Ensure to clearly write question number before the answer the question.
3. Worked out solutions on supplements are must for some problems.
4. Please ensure to write Question number in a box as a heading to the upcoming answer on supplements e.g.

Question 1

===

1) Which one of the following is a best match for a Taylor series: (1 mark)

a) $\sum_{n=0}^{\infty}\frac{(\Delta x)^n}{n!}(y^{(n)}(x_0))$

b) $\sum_{n=0}^{\infty}\frac{(x)^n}{n!}$

c) $\sum_{n=0}^{\infty}(x)^n$

d) $\sum_{n=0}^{\infty}\frac{(\Delta x)^{n-1}}{(n-1)!}(y^{(n)}(x_0))$

(note: $y^{(n)}(x_0)$ denotes n'th derivative of the y evaluated at $x = x_0$)

2) An investor puts Rs. 10000 in fixed deposit scheme where interest is compounded almost at every instant at the rate of interest 6% per year for 5 years. Then at the end the 2nd year he invests additional Rs.5000 in the same scheme in the accrued amount.

a) Can you write a single differential equation that models above scenario (please clearly mention the meaning of the variables)? (1 mark)

b) What is the solution to the differential equation and final amount accumulated at the end of 5 years? (1 mark)

3) For given ODE please answer

a) Solve: $\frac{dy}{dx} = 2 - 4y$ with $y(0) = 3$ (1 mark)

b) For above ODE problem, $y(\infty) = ?$ (1 mark)

4) Find complete solution for:

a) $y' - 8y = 2H(t-3)$ (1 mark)

b) $y' - 8y = e^{2t}$ (1 mark)

5) What are values of M and N in case of proposed solution $y = Mcos(\omega t) + Nsin(\omega t)$ for ODE given by $y' - 5y = \cos(4t) + 2\sin(4t)$? (2 mark)

6) What is R and ϕ in following polar form conversion:

$$\cos(2t) + \sin(2t) = Rcos(2t - \phi)$$ (2 mark)

7) What is α and G in the $Ge^{-i\alpha} = \frac{1}{re^{i\alpha}}$ where r and α correspond to the polar form

$re^{i\alpha} = \sqrt{3}i + 1$ (2 mark)

8) What are the i) order, ii) degree, iii) Homogeneous/Non-homogeneous, iv) Linear/non-linear attributes corresponding to following two differential equations:

a) $(y'')^2 + (y')^3 - 3y - \cos(x) = 0$ (1 mark)

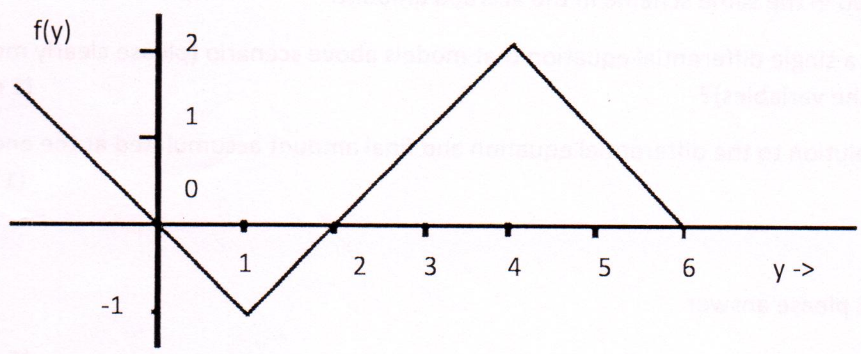
b) $(y'')^2 - 3y'''y^3 = 0$ (1 mark)

9) For the $y' = f(y)$ equation where $f(y)$ is given by following plot, estimate $y(\infty)$ if

a) $y(0) = 1.01$ (1 mark)

b) $y(0) = 2.01$ (1 mark)

c) $y(0) = 5.01$ (1 mark)



10) Does following ODE satisfies exactness? $(1 - 2xy^2)dx - (2x^2y)dy = 0$ (2 marks)