IIT DHARWAD
AUTUMN 2019

Engg. Mathematics for Advanced Studies

ODE - Qz - Retest - Solutions

Q.1

Given, $y' = 2 - y$

Key understanding → for stable equilibrium

$$y' = 0 \quad \& \quad \cancel{y'' < 0} \quad \frac{d}{dy}(y') < 0$$

Note $y' = \frac{dy}{dt}$ if $y = y(t)$

Answer → ⓐ $y = 2$ is stable equilibrium.

Q.2

ⓐ $y' = y\sin(t)$ False - Not autonomous eqn.

ⓑ $yy' = \sin(e^y)$ False - Not 2^{nd} order ODE

ⓒ $yy' = \sin(e^y)$ True - Autonomous eqn. $y' = f(y)$.

ⓓ $(y')^2 - 3y\sin(t) = t - 1$ False - $(y')^2$ is non-linear.

ⓔ $y' = by^4$ True - Bernoulli Eqn. $y' + p(t)y = g(t)\cdot y^a$

ⓕ $y' = bx^4$ True

ⓖ $y' + 3\log y = x$ False - Not linear due to "$\log y$"

ⓗ $y' = \frac{t}{y^2}$ True - However complete sol^n. $\frac{1}{3}y^3 = \frac{1}{2}t^2 + \text{const.}$

Q.3

Bernoulli eqn. $y' + p(t)y = g(t)\cdot y^{a}$

It is a Non-linear equation. In general, non-linear diff. eqn. are difficult to solve. However, with Bernoulli eqn. we can change/substitute variable to bring it to a linear form. That magical substitution is

$$u = y^{1-a}$$

Linearized version: $u' + (1-a)pu = (1-a)g$

Q.4

$y' - 2y = \delta(2)$, $y(0) = 1$

There is typo in this question. It is retracted and all those who attempted / not attempted this question but indicated mistake will get 2 Marks. (Full score).

Corrected Question → $y' - 2y = \delta(t-2)$, $y(0) = 1$

Null solutions $y' - 2y = 0 \Rightarrow y_n = ke^{2t}$ at $y(0) = 1$, $\Rightarrow k = 1$

Particular solution $y_p = e^{at}\int_0^t e^{-as}\cdot q(s)\, ds$ ∵ $q(s) = \delta(t-2)$

$= e^{a(t-2)}$ ...... (for $2 < t$) (for $2 > t \to y_p = 0$)

$y = y_n + y_p = e^{2t} + e^{2(t-2)}\cdot H(t-2)$

Note: $\delta(t)$ is equivalent to $y(0) = 1$ for $y' = ay$
similarly $\delta(t-T)$ is equivalent to $y(T) = 1$ for remaining time $(t-T)$

Q. 5

$$y' - \sqrt{3}y = \cos(t) + \sin(t) \quad \text{with } y(0) = 1$$

Method of assuming $y = A\cos(t) + B\sin(t)$ will not help here since initial condition can ~~dedtate~~ force constants to take up wrong values. especially since we are ignoring any constants in y. i.e. $y = A\cos(t) + B\sin t + K$.

We apply more structured approach using complex numbers.

$\boxed{\text{Step 1}}$ – get RHS in $R\cos(\omega t - \phi)$ form.

$\cos(t) + \sin(t) \Rightarrow A\cos(\omega t) + B\sin(\omega t)$ with

$A = 1 \quad \omega = 1 \quad B = 1$

$\therefore R = \sqrt{A^2 + B^2} = \sqrt{2} \qquad \phi = \tan^{-1}\left(\frac{B}{A}\right) = \frac{\pi}{4}$

$$\therefore y' - \sqrt{3}y = \cancel{\sqrt{2}}\ \sqrt{2}\cos\left(t - \frac{\pi}{4}\right)$$

$\boxed{\text{Step 2}}$ Find ~~equivalent~~ counterpart problem in complex field.

$$y_c' - \cancel{\sqrt{3}}^{\,a} y_c = Re^{i\omega t}$$

Note $\sqrt{2}\cos\left(t - \frac{\pi}{4}\right)$ is real part of $\sqrt{2}\, e^{i\left(t - \frac{\pi}{4}\right)}$

Hence

$$y_c' - \sqrt{3}y_c = \sqrt{2}\, e^{i\left(t - \frac{\pi}{4}\right)}$$

Real part of the solution y_c which will solve above complex ODE will be the solution.

Solution for $y_c' - ay_c = R_1 e^{i(\omega t - \phi)}$ is

$$y_p(t) = \frac{R_1}{i\omega - a} e^{i(\omega t - \phi)} = R_1\left(\frac{1}{re^{i\alpha}}\right) e^{i(\omega t - \phi)}$$

where $re^{i\alpha}$ is polar form of $i\omega - a$.

P04

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$$r = \sqrt{\omega^2 + a^2} = \sqrt{1 + (\sqrt{3})^2} = 2$$

$$\alpha = \tan^{-1}\left(\frac{-\omega}{a}\right) = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$$

$$\therefore\ y_{cp}(t) = \sqrt{2}\left(\frac{1}{2e^{i\alpha}}\right)e^{i(t-\frac{\pi}{4})}$$

$$y_{cp.} = \frac{1}{\sqrt{2}}e^{i(t-\frac{\pi}{4}-\alpha)} \quad \text{where } \alpha = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$$

$$\therefore\ \quad = \frac{1}{\sqrt{2}}\left[\cos\left(t-\frac{\pi}{4}-\alpha\right) + i\sin\left(t-\frac{\pi}{4}-\alpha\right)\right]$$

$$\therefore\ \boxed{y_{real} = \frac{1}{\sqrt{2}}\cos\left(t-\frac{\pi}{4}-\alpha\right)}$$

$$y = y_N + y_p = ke^{\sqrt{3}t} + \frac{1}{\sqrt{2}}\cos\left(t-\frac{\pi}{4}-\alpha\right)$$

$$\text{where } \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = \frac{2\pi}{3}$$

Given $y(0) = 1$

$$1 = k + \frac{1}{\sqrt{2}}\cos\left(-\frac{\pi}{4}-\alpha\right) \Rightarrow k = 1 - \frac{1}{\sqrt{2}}\cos\left(\frac{\pi}{4}+\alpha\right)$$

$$= 1 - \frac{1}{\sqrt{2}}\cos\left(\frac{\pi}{4}+\frac{2\pi}{3}\right)$$

$$\therefore\ = 1 - \frac{1}{\sqrt{2}}\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{2\pi}{3}\right) + \frac{1}{\sqrt{2}}\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{2\pi}{3}\right)$$

$$= 1 - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\left(-\frac{1}{2}\right) + \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}$$

$$= 1 + \frac{1}{4} + \frac{\sqrt{3}}{4} = \frac{5+\sqrt{3}}{4}.$$

$$\boxed{k = \frac{5+\sqrt{3}}{4}}$$

$\boxed{Q.6}$

PO5

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$$\frac{dT}{dt} = k(T_\infty - T) \Rightarrow \frac{dT}{T_\infty - T} = k\,dt \Rightarrow \frac{dT}{T - T_\infty} = -k\,dt.$$

$$\therefore \ln(T - T_\infty) = \cancel{e^{-k}} -kt + C_0$$

$$\therefore T - T_\infty = e^{-kt + C_0}$$

Given. at $t=0$ $\quad T - T_\infty = 80°C \Rightarrow e^{C_0} = 80$

at $t = 90$ $\quad T - T_\infty = 20°C \Rightarrow \cancel{e^{-k90+C_0}}$

$$e^{-90k + C_0} = 20$$

i.e. $$e^{C_0} \cdot e^{-90k} = 20$$

$$\therefore e^{-90k} = 0.25$$

$$\therefore k = +0.01540$$

$\therefore$ For $T - T_\infty = 10$ $\qquad 10 = e^{C_0} \cdot e^{-kt} = 80e^{-kt} = 80e^{-0.0154t}$

$\therefore$ Hence $t = \dfrac{\ln\left(\frac{10}{80}\right)}{-0.0154} = 135$ mins

$\therefore$ Ans. 135 min

$\boxed{\text{Extra.}}$

For Alternative Soln method for Q.5.

$$y' - \sqrt{3}y = \cos(t) + \sin(t) \quad \text{with } y(0) = 1.$$

Null soln. $y_H = \cancel{k}\, e^{\sqrt{3}t}$.

Guess soln. (Because RHS has only sin & cos & LHS has y & y')

$$y = A\cos(\omega t) + B\sin(\omega t) + y_N$$

i.e. $A = 1$, $B = 1$, $\omega = 1$, $y_N = ke^{\sqrt{3}t}$

$$y' - \sqrt{3}y = \left(-A\omega\sin(\omega t) + B\omega\cos(\omega t) + \sqrt{3}ke^{\sqrt{3}t}\right)$$
$$- \sqrt{3}\left(A\cos(\omega t) + B\sin(\omega t) + ke^{\sqrt{3}t}\right)$$
$$= (-\sqrt{3}B - A\omega)\sin(\omega t) + (B\omega - \sqrt{3}A)\cos(\omega t) + \sqrt{3}ke^{\sqrt{3}t} - \sqrt{3}ke^{\sqrt{3}t}$$
$$= \cancel{-A\omega}\,(B\omega - \sqrt{3}A)\cos(\omega t) + (-\sqrt{3}B - A\omega)\sin(\omega t).$$

given Diff. eqn. RHS can be compared w/ above eqn RHS.

$\therefore$ $B - \sqrt{3}A = 1 \Rightarrow B = 1 + \sqrt{3}A$

$A + \sqrt{3}B = -1 \Rightarrow A + \sqrt{3}(1 + \sqrt{3}A) = -1 \Rightarrow A = \dfrac{-1-\sqrt{3}}{4}$

$B = \dfrac{1-\sqrt{3}}{4}$

$$\therefore \quad y = \left(\frac{-1-\sqrt{3}}{4}\right)\cos(t) + \left(\frac{1-\sqrt{3}}{4}\right)\sin(t) + ke^{\sqrt{3}t}.$$

since $y(0) = 1 \Rightarrow \cancel{k=}\ 0 = \left(\dfrac{-1-\sqrt{3}}{4}\right) + 0 + k$

i.e. $k = \dfrac{1+\sqrt{3}}{4}$

$$\therefore \boxed{y = \left(\frac{-1-\sqrt{3}}{4}\right)\cos(t) + \left(\frac{1-\sqrt{3}}{4}\right)\sin(t) + \left(\frac{1+\sqrt{3}}{4}\right)e^{\sqrt{3}t}}$$