IIT DHARWAD.

ENGG. MATHS FOR ADV. STUDIES

P-01
ODE-A01-Soln

ODE - Assignment 01 - Solutions

Q.1

Sr. No.	Equation	Order (N)	Degree (N)	L/NL	H/NH
1.	$y''' = e^x$	3	1	L	NH
2.	$y'' + 2y' + 2y = 0$	2	1	L	H
3.	$x + yy' = 0$	1	1	NL	NH
4.	$y'/x + x = 0$	1	1	L	NH
5.	$y' + \sin(y) = x$	1	1	NL	NH
6.	$(y'')^3 + \log(y) = x$	2	3	NL	NH
7.	$y'' + 3yy' + y = \cos(t)$	2	1	NL	NH
8.	$y'' + (y')^2 + 4y = 2x$	2	1	NL	NH
9.	$y'' + (y')^2 + 4y = e^x$	2	1	NL	NH
10.	$y'' + (y')^2 + 4y = 0$	2	1	NL	H

Q.2

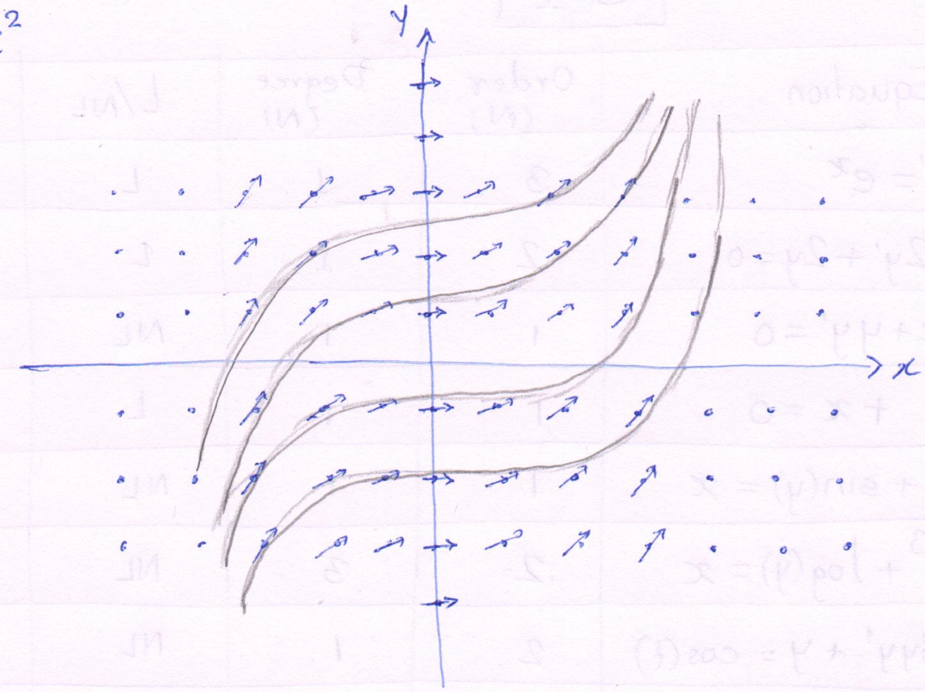
① draining out water from tank. Rate of flux is decreasing with the available pressure head.

② Cooling time for tea / eatables.

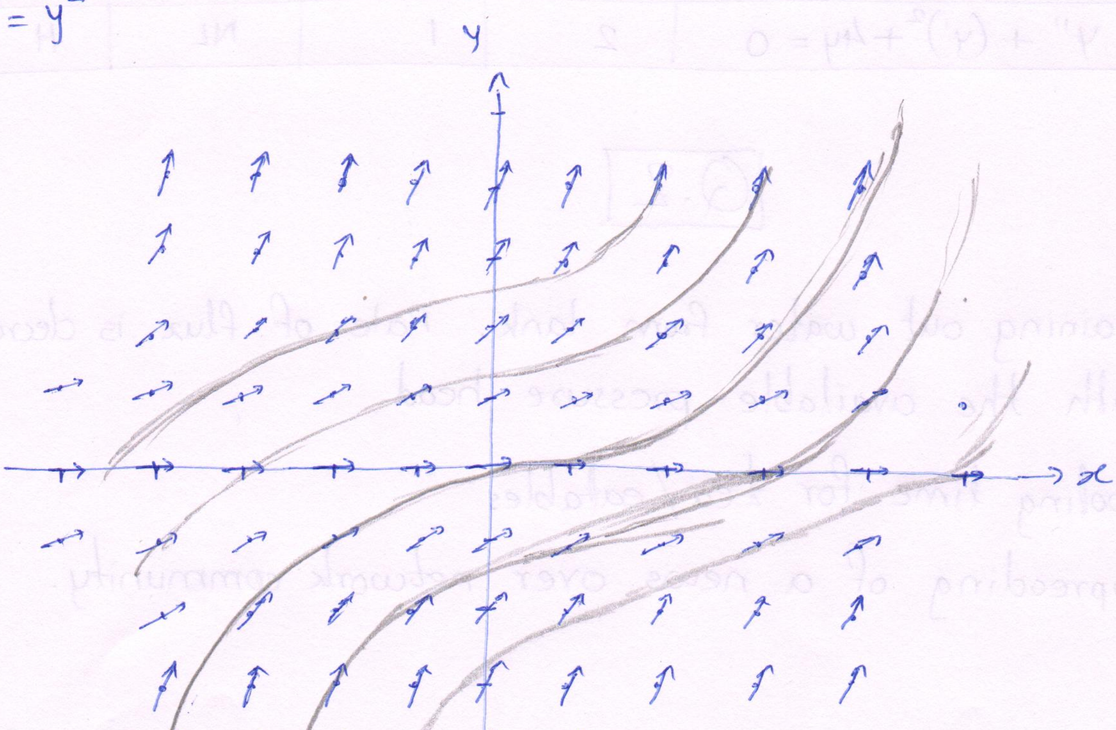
③ Spreading of a news over network community.

Q.3

a) $y' = x^2$



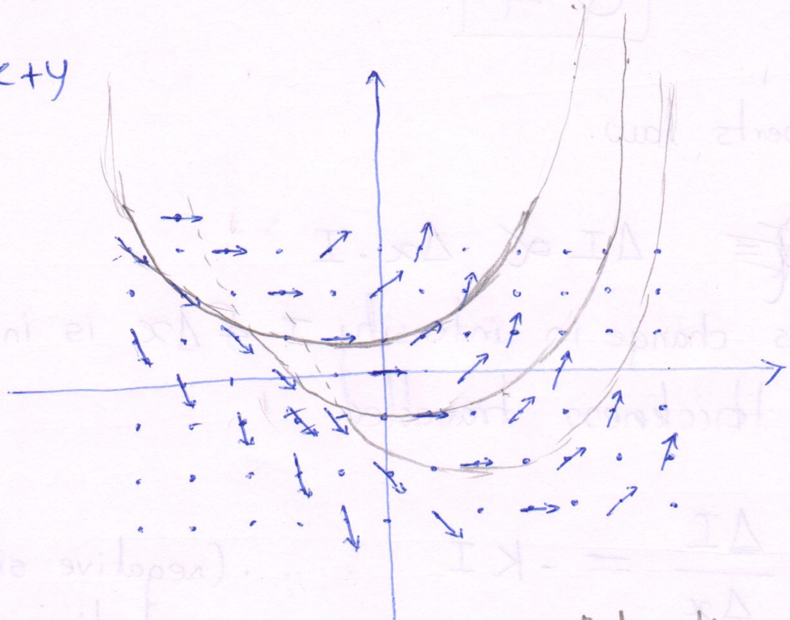
b) $y' = y^2$



NOTE → Each curve is offset of previous curve (along x dir)

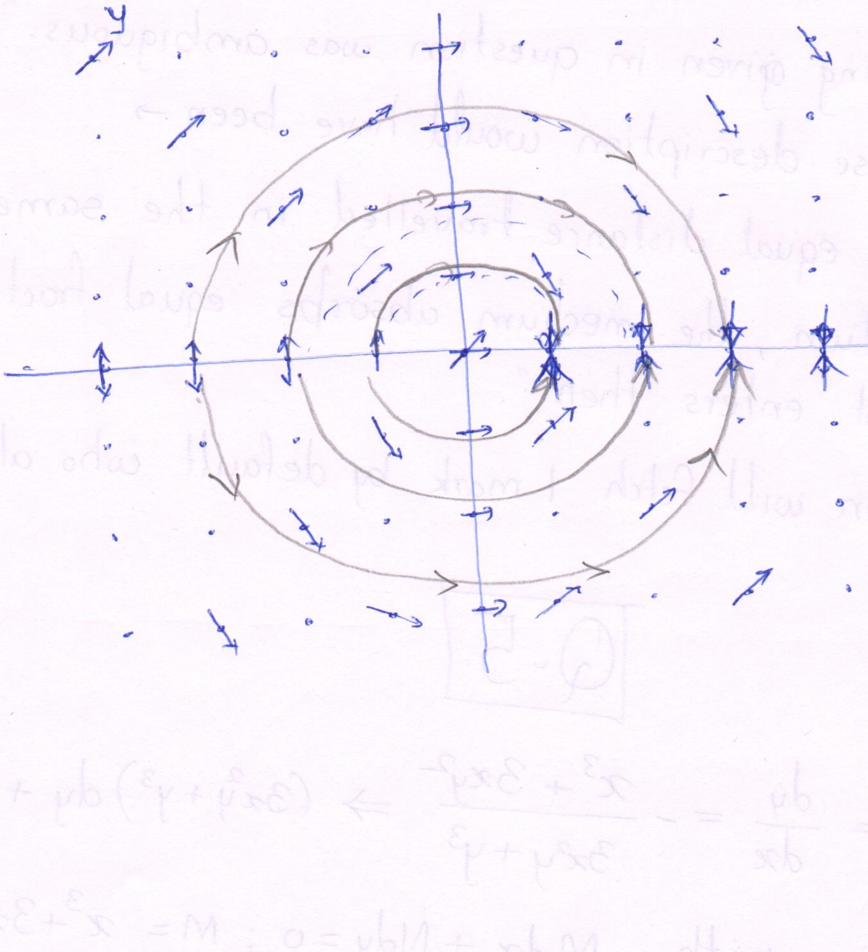
Q.3 contnd.

c) $y' = x + y$



NOTE → each curve is offset in $-45°$ direction.

d) $y' = \frac{-x}{y}$



Q.4

This is Lamberts law.

$$\cancel{\Delta Q} \equiv \quad \Delta I \propto \Delta x \cdot I$$

where ΔI is change in intensity I & Δx is increament inthe thickness travelled.

$$\therefore \quad \frac{\Delta I}{\Delta x} = -KI \quad \ldots$$ (negative sign as ΔI is reduction in intensity)

Taking limits

$$\frac{dI}{dx} = -kI$$

(Note → Wording given in question was ambiguous. More precise description would have been →
" For equal distance travelled in the same absorbing medium, the medium absorbs equal fraction of light that enters them".
This question will fetch 1 mark by default who attempted it)

Q.5

$$y' = \frac{dy}{dx} = -\frac{x^3 + 3xy^2}{3x^2y + y^3} \Rightarrow (3x^2y + y^3)\,dy + (x^3 + 3xy^2)\,dx = 0.$$

Comparing with $M\,dx + N\,dy = 0$; $M = x^3 + 3xy^2$

$N = 3x^2y + y^3$

Exactness condition: $\frac{\partial M}{\partial y} = 6xy$ & $\frac{\partial N}{\partial x} = 6xy$.

$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ i.e. exactness condition Satisfied.

Q.5 contnd.

Exactness condition satisfied means there exists u such that

$$M = \frac{\partial u}{\partial x} \quad \& \quad N = \frac{\partial u}{\partial y}$$

i.e. given ~~integ~~ Diff. eqn. becomes $\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0.$

i.e. $\frac{du}{dv(x,y)} = 0$

i.e u = constant.

$$M = \frac{\partial u}{\partial x} \Rightarrow \quad u = \int M dx + f_1(y) = \int (x^3 + 3xy^2) dx + f_1(y)$$

$$u = \left[\frac{x^4}{4} + \frac{3}{2}x^2y^2\right] + f_1(y) \quad \text{— (A)}$$

$$N = \frac{\partial u}{\partial y} \Rightarrow \quad u = \int N dy + f_2(x) = \int (3x^2y + y^3) dy + f_2(x)$$

$$u = \left[\frac{3}{2}x^2y^2 + \frac{y^4}{4}\right] + f_2(x) \quad \text{— (B)}$$

Comparing expression for u in (A) & (B), we can combinedly write,

$$u = \frac{x^4}{4} + \frac{3}{2}x^2y^2 + \frac{y^4}{4} \; (= \text{Const.})$$

$$\because u = \text{const} \Rightarrow \boxed{x^4 + 6x^2y^2 + y^4 = \text{const.}}$$

(b) This is implicit solution where we can check if given (x_1, y_1) pair satisfieds eqn. or it does not.
If explicit, equation reports y value (or x value) for a given x value explicitly (for y value).

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Q. 6

As per newton's law of cooling

$$\frac{d}{dt}(T - T_\infty) = -k(T - T_\infty)$$

i.e. $$\frac{dT}{dt} = -k(T - T_\infty)$$ .. where $T_\infty \to$ ambient temp. & T is current temp. at current time t.

$\therefore$ $$\frac{dT}{(T - T_\infty)} = -k\,dt$$

$\therefore$ $$\ln(T - T_\infty) = -kt + c_0 \Rightarrow T - T_\infty = e^{(-kt + c_0)} = e^{c_0} \cdot e^{-kt}$$

At time $t = 0$, $T_0 = 150 \Rightarrow 150 - 21 = e^{c_0} \cdot e^{0} = e^{c_0}$

$\therefore e^{c_0} = 129$.

$t = 10$, $T_{10} = 95 \Rightarrow 95 - 21 = e^{c_0} \cdot e^{-k(10)}$

$\therefore 74 = 129\, e^{-10k}$

$\therefore k = 0.055574$

Since, $$T(t) = T_\infty + e^{c_0} \cdot e^{-kt}$$

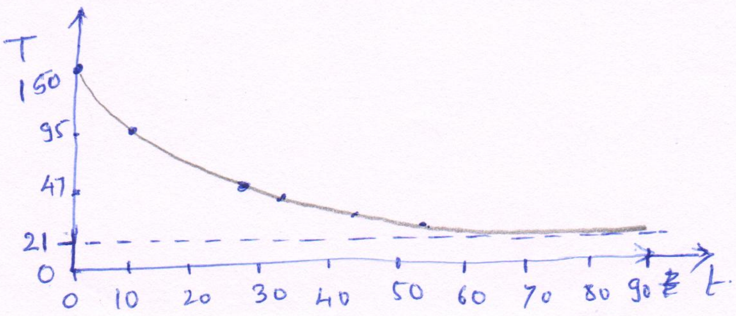
(a) $$T(t) = 21 + 129 \cdot e^{-(0.055574)t}$$ ('t' in minutes)

or $$t = \frac{-1}{0.055574} \ln\left(\frac{T(t) - 21}{129}\right)$$ (T in Degree Celcius)

(b)

T	150	95	47	42	32	27	24	23	22
time	0	10	28.82	32.66	44.3	55.207	67.68	74.98	87.45

(c)



Q.7

RL circuit $\Rightarrow$ Take the same example with $R = 5\,\Omega$, $L = 0.1$ Henry
$E(t) = 12$ Volt $=$ const.

$$L\frac{dI}{dt} + RI = E(t).$$

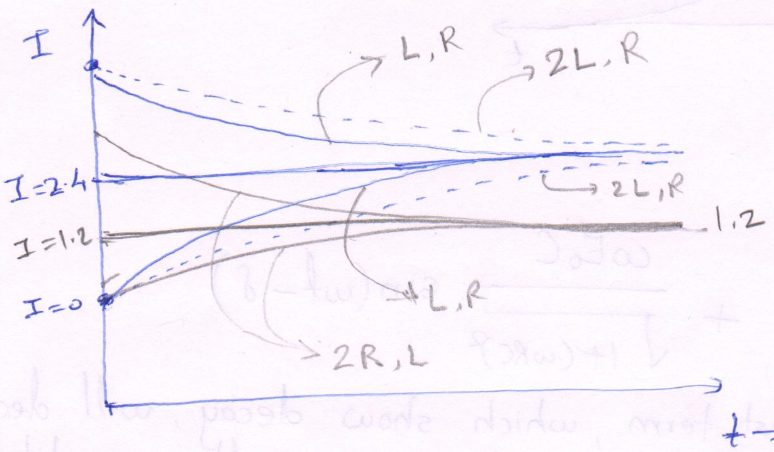
For const. EMF

given default $\Rightarrow (0.1)\frac{dI}{dt} + 5I = 12 \Rightarrow \frac{dI}{dt} + 50I = 120 \Rightarrow I = 2.4 + ce^{-50t}$ — (7.1)

with $2R \Rightarrow (0.1)\frac{dI}{dt} + 10I = 12 \Rightarrow \frac{dI}{dt} + 100I = 120 \Rightarrow I = 1.2 + ce^{-100t}$ — (7.2)

with $2L \Rightarrow (0.2)\frac{dI}{dt} + 5I = 12 \Rightarrow \frac{dI}{dt} + 25I = 60 \Rightarrow I = \frac{60}{25} + ce^{-25t}$

$= 2.4 + ce^{-25t}$ — (7.3)



For sinusoidal EMF

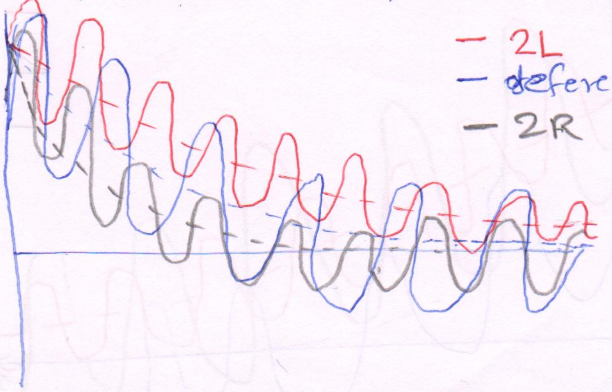
eqn. (6) $\Rightarrow I(t) = ce^{-(R/L)t} + \frac{E_0}{\sqrt{R^2 + \omega^2 L^2}}\sin(\omega t - \delta)$.

If R or L is doubled, clearly, the second term will have smaller ~~app~~ amplitude

If $R \to 2R$ then first term, which shows decay, will decay fast.

If $L \to 2L$ then second term, which shows decay, will decay slowly.

Hence



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Q.7 Continued

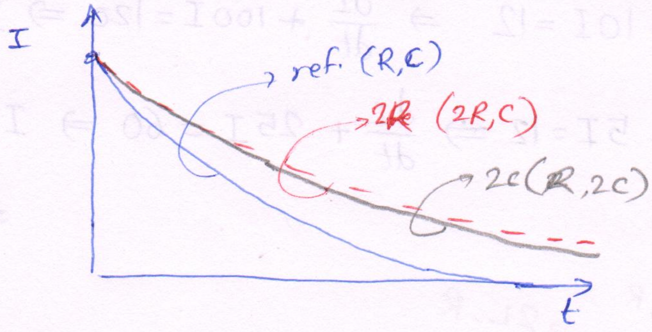
Looking at eqn. for $I(t)$ that solves RC circuit, $\left(R\frac{dI}{dt} + \frac{1}{C}I = \frac{dE}{dt}\right)$

$$I(t) = e^{-\left(\frac{t}{RC}\right)} \cdot \frac{1}{R} \int e^{\frac{t}{RC}} \frac{dE}{dt}\, dt + c$$

For constant electromotive for $E(t) = E$ i.e. $\frac{dE}{dt} = 0$.

$$I(t) = c\, e^{\frac{-t}{RC}}$$

doubling either of R or C will cause slower decay

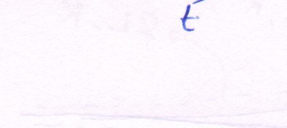


For sinusoidal EMF.

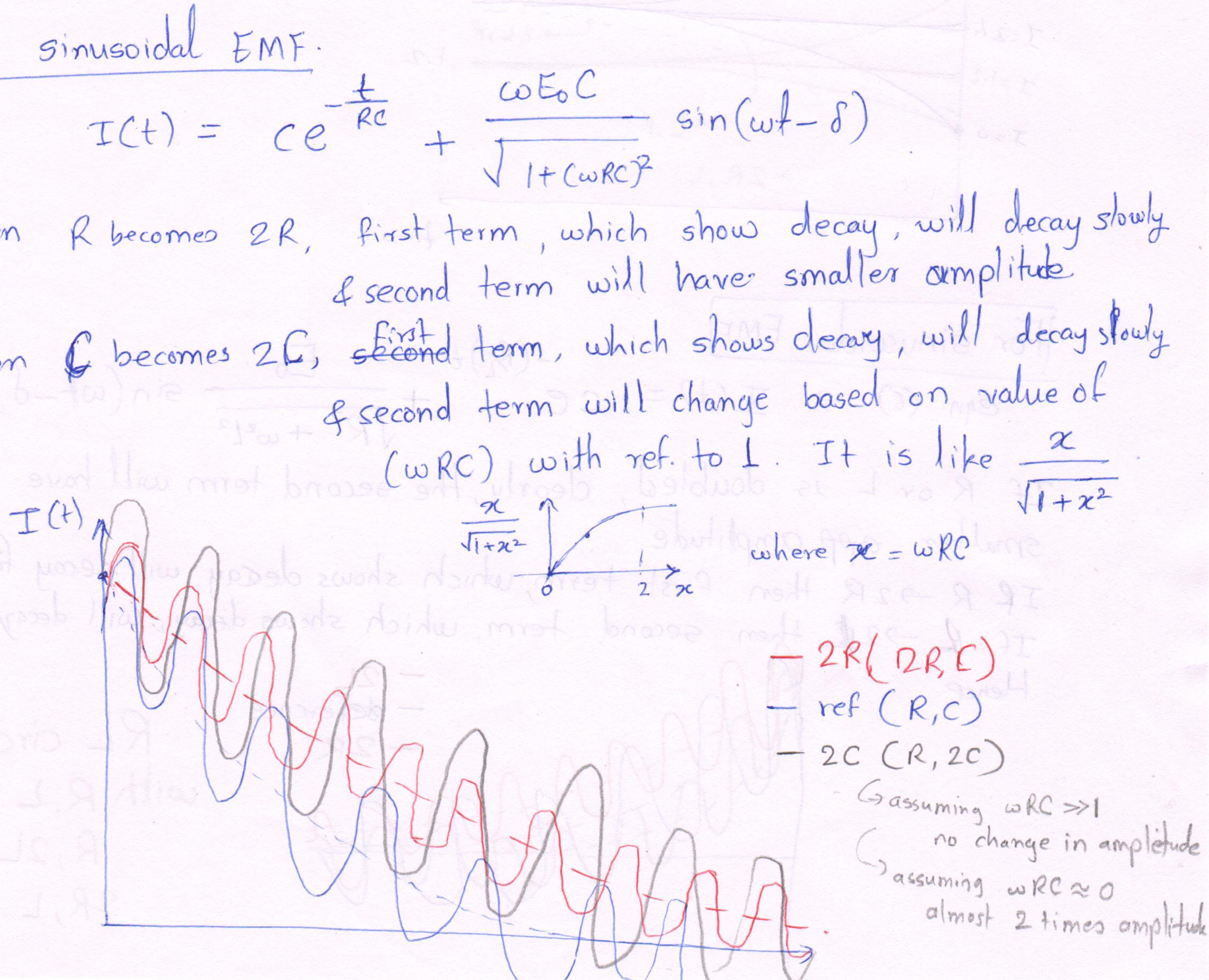
$$I(t) = c e^{-\frac{t}{RC}} + \frac{\omega E_0 C}{\sqrt{1+(\omega RC)^2}} \sin(\omega t - \delta)$$

when R becomes 2R, first term, which show decay, will decay slowly & second term will have smaller amplitude

when C becomes 2C, ~~second~~ first term, which shows decay, will decay slowly & second term will change based on value of (ωRC) with ref. to 1. It is like $\frac{x}{\sqrt{1+x^2}}$

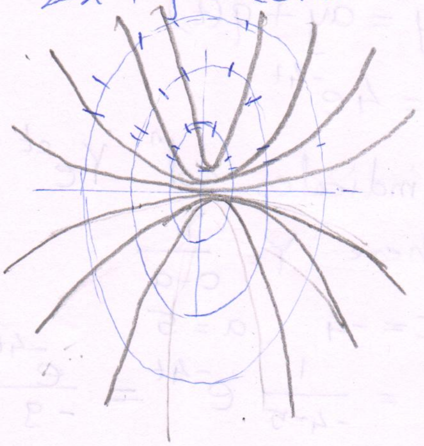


where $x = \omega RC$



Q. 7

(e) $2x^2 + y^2 = \text{const} \rightarrow \frac{x^2}{1} + \frac{y^2}{2} = \text{const.}$ i.e. ellipse.



Similar to fig. 28 in Kreyszig.

Q. 8

We will explore $e^{at} \cdot e^{at}$.

(a) $y' = (a+b)y \Rightarrow \frac{dy}{y} = (a+b)\,dt \Rightarrow \ln(y) = (a+b)t + c$

$\Rightarrow y = e^{(a+b)t + c}$

(b) Assume $y = e^{at} e^{bt}$.

$$y' = \frac{d}{dt}(e^{at}) \cdot e^{bt} + e^{at} \cdot \frac{d}{dt}(e^{bt})$$

$$= a e^{at} \cdot e^{bt} + b e^{at} \cdot e^{bt}$$

$$= (a+b) e^{at} e^{bt}$$

$$= (a+b) y \longrightarrow \text{Verified successfully.}$$

(NOTE the sequence of diffⁿ. it is the key point in this example).

(c) For $e^{(A+B)} = e^{At} \cdot e^{Bt} \Rightarrow$ with A & B being matrices.

assume $y = e^{At} \cdot e^{Bt}$ solves $y' = (A+B)y$.

$$y' = \frac{d}{dt}(e^{At} \cdot e^{Bt}) = \frac{d}{dt} e^{At} \cdot e^{Bt} + e^{At} \cdot \frac{d}{dt} e^{Bt} \ldots$$

$$= A e^{At} \cdot e^{Bt} + \underbrace{e^{At} \cdot B e^{Bt}}$$

this being matrix product $e^{At} \cdot B e^{Bt}$ not necessary same at $B \cdot e^{At} e^{Bt}$ which could have completed proof.

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Q.9

ⓐ $y' = 5y + 4e^{-4t}$ $\Rightarrow$ std. format $y' = ay + q(t)$

$a = 5, \quad q(t) = 4e^{-4t}$

$\hookrightarrow$ indicates sol^n Ye^{ct}

where $Y = \dfrac{1}{c-a}$

with $c = -4 \quad a = 5$

Particular sol^n $= \dfrac{1}{-4-5} e^{-4t} = \dfrac{e^{-4t}}{-9}$

Null solution for $y' = 5y \Rightarrow y_N = e^{5t} \cdot \cancel{y(0)} k_1$

$$\therefore \quad y = k_1 \cdot e^{5t} - \frac{1}{9} e^{-4t}$$

ⓑ $y' = 5y + e^{-5t}$

$$y = k_1 e^{5t} + \frac{1}{-5-5} e^{-5t} = k_1 e^{5t} - \frac{1}{10} e^{-5t}$$

ⓒ $y' = 5y + e^{5t}$

This is critical case where $Y = \dfrac{1}{c-a} = \dfrac{1}{5-5} = \dfrac{1}{0}$!

Particular solution here is $t \cdot e^{at}$

$$\therefore \quad y = k_1 e^{at} + t \cdot e^{at}$$

for $t = 0$ second term is zero $\Rightarrow y(0) = k_1$

$$\therefore \quad y = y(0) e^{at} + t e^{at}$$

$$y = y(0) e^{5t} + t \cdot e^{5t}$$

$\boxed{Q.9}$

ⓓ $y' = 3y + 5e^{2it}$

form $q(t) = Re^{i\omega t}$ with $a = 3$, $R = 5$, $\omega = 2$ $\Rightarrow$ Particular solution $Y = \frac{R}{j\omega - a}$ & $y = Ye^{i\omega t}$

$\therefore \quad y = k_1 e^{3t} + \frac{5}{2i - 3} e^{i2t}$

k_1 to be determined based on boundary cond. (for all other cases as well in this Q.9).

ⓔ $y' = 2y + \underbrace{3\cos(t) + 4\sin(t)}_{\downarrow}$

$\underbrace{A\cos(t) + B\sin(t)}_{\downarrow}$

$R = \sqrt{A^2 + B^2}$ & $\phi = \tan^{-1}\left(\frac{B}{A}\right)$ $\quad \omega = 1$

$\therefore \quad R = 5 \quad$ & $\phi = \tan^{-1}\left(\frac{4}{3}\right)$

$\therefore \quad 3\cos(t) + 4\sin(t) = R \frac{\cos}{\sin}(\omega t - \phi)$

$= 5 \frac{\cos}{\sin}(t - \phi)$

$\therefore \quad y' = 2y + 5\cos(\overset{\omega=1}{t} - \phi)$

Null sol^n $\quad y' - 2y = 0 \Rightarrow y = k_1 e^{2t}$

$y' - 2y = 5\cos(t - \phi)$ is real part of the equivalent complex solution $y_c' - 2y_c = Re^{i(\omega t - \phi)}$, $\omega = 1$

$= 5e^{i(t-\phi)}$

i.e. real $\left[R \cdot \frac{1}{\sqrt{\omega^2 + a^2}} \cdot e^{i(\omega t - \phi - \alpha)}\right]$

i.e. real $\left[5 \cdot \frac{1}{\sqrt{1+4}} \cdot e^{i(t - \phi - \alpha)}\right]$ where $\alpha = \tan^{-1}\left(\frac{\omega}{-a}\right) = \tan^{-1}\left(\frac{-1}{2}\right)$

i.e. $y = \sqrt{5}\cos(t - \phi - \alpha)$ with $\phi = \tan^{-1}\left(\frac{4}{3}\right)$ & $\alpha = \tan^{-1}\left(\frac{-1}{2}\right)$.

Q. 10

$$\sin(5t) + \cos(5t) = R\cos(\omega t - \phi)$$

$A = 1 \quad B = 1 \quad \omega = 5 \Rightarrow R = \sqrt{A^2 + B^2} = \sqrt{2}$

$$\phi = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

$$\therefore \sin(5t) + \cos(5t) = \sqrt{2}\cos\left(5t - \frac{\pi}{4}\right).$$

Q. 11

(a)

$$\frac{dp}{dt} = \frac{a}{100} \cdot p$$

(b)

$$\ln(p) = \frac{a}{100}t + c \Rightarrow p = e^{\left(\frac{at}{100}\right)} \cdot e^{\text{const.}}$$

at $t = 0$ $\quad p(0) = p_0 \Rightarrow \quad p_0 = e^{\text{const.}} \Rightarrow \quad p(t) = p_0 e^{\left(\frac{at}{100}\right)}$

(c) For doubling p_0 to $2p_0$ at time T_D $\quad 2p_0 = p_0 e^{\left(\frac{aT_D}{100}\right)}$

$$\therefore \frac{aT_D}{100} = \ln 2$$

$$\therefore aT_D = 100 \ln 2 = 69.3147.$$

i.e. (Rate of interest in %) $\times$ (Time to double) $= 69.31$

Rule of 72 $\rightarrow$ same rule approximation for quick calculation of doubling time at a given interest rate a. ($\therefore a \times T_D = 72$)

(d) $p(t) = p_0 e^{\left(\frac{at}{100}\right)}$ gave 69.31 because $\Delta t \to 0$ and p is updated frequently. In practice its not the case with bank. hence Rule of 72 has $72 > 69.31$. because compounding is quarterly or annually. causing a bit extra time for doubling