Solutions to Assignment 01

MODULE: (Numerical Methods)

COURSE -: ENGG. MATHEMATICS FOR ADVANCED STUDIES

Q.1

a) $h_i = \Delta_i = 2$ (since time variable is consistently increasing by 2 from one reading to next)

b) It is prescribed to define a cubic spline with free run out condition. $g''(x_0)$ & $g''(x_7)$ are the second derivatives at the endpoints.

Hence,

$$g''(x_0) = g''(x_7) = 0 \quad \ldots (\because \text{Free run out})$$

$$\therefore \quad a = b = 0$$

c)

Eqn. 1.7 in Prof. Parviz Moin's book can be reduced to following simpler version taking into account the fact that $\Delta_{i-1} = \Delta_i = \Delta_{i+1} = \ldots = h \,(=2)$.

$$\frac{h}{6} g''(x_{i-1}) + \frac{2h}{3} g''(x_i) + \frac{h}{6} g''(x_{i+1})$$

$$= \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h} \quad \text{— (A)}$$

$$\frac{h^2}{6} g''(x_{i-1}) + \frac{2h^2}{3} g''(x_i) + \frac{h^2}{6} g''(x_{i+1})$$

$$= f(x_{i+1}) - 2f(x_i) - f(x_{i-1}) \quad \text{— (B)}$$

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(c)

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{h}{6} & \frac{2h}{3} & \frac{h}{6} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{h}{6} & \frac{2h}{3} & \frac{h}{6} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{h}{6} & \frac{2h}{3} & \frac{h}{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{h}{6} & \frac{2h}{3} & \frac{h}{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{h}{6} & \frac{2h}{3} & \frac{h}{6} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{h}{6} & \frac{2h}{3} & \frac{h}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} g''(x_0) \\ g''(x_1) \\ g''(x_2) \\ g''(x_3) \\ g''(x_4) \\ g''(x_5) \\ g''(x_6) \\ g''(x_7) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(0 \\ \frac{1}{2}(\\ \\ \\ \\ \\ \\ \\ \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 0 \\ (0.12507) - 2(-0.12828) + (-0.10022) \\ (-0.12828) - 2(-0.10022) + (0.66500) \\ (-0.10022) - 2(0.66500) + (0.66500) \\ (0.66500) - 2(\cancel{-0.10022}\ 0.66500) + (\cancel{-0.12828}\ -0.10022) \\ (\cancel{-0.10022}\ 0.6650) - 2(\cancel{-0.12828}\ -0.10022) + (\cancel{0.12507}\ 0.12828) \\ (\cancel{-0.12828}\ -0.10022) - 2(\cancel{0.12507}\ 0.12828) + (0.12507) \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0.0 \\ 0.28141 \\ 0.73716 \\ -0.76522 \\ -0.76522 \\ 0.73716 \\ 0.28141 \\ 0 \end{bmatrix}$$

Solving above system of linear algebraic eqns.
(See Appendix for MATLAB/OCtave code)

$g''_0 = 0.0$	$x_0 = 0.0$	$V_0 = 0.12507$
$g''_1 = 0.01892$	$x_1 = 2.0$	$V_1 = -0.12828$
$g''_2 = 0.34642$	$x_2 = 4.0$	$V_2 = -0.10022$
$g''_3 = -0.29885$	$x_3 = 6.0$	$V_3 = 0.66500$
$g''_4 = -0.29885$	$x_4 = 8.0$	$V_4 = 0.66500$
$g''_5 = 0.34642$	$x_5 = 10.0$	$V_5 = -0.10022$
$g''_6 = 0.01892$	$x_6 = 12.0$	$V_6 = -0.12828$
$g''_7 = 0.0.$	$x_7 = 14.0$	$V_7 = 0.12507$

(d)

Above information, along with given datapoints can be used to formulate 7 functions for 7 intervals. i.e. $g_i(x)$ for $i=1,7$

e.g. For segment 2 i.e. $i=2$

$$g_2(x) = \frac{g''_i}{6}\left[\frac{(x_{i+1}-x)^3}{\Delta_i} - \Delta_i(x_{i+1}-x)\right].$$

$$+ \frac{g''_{i+1}}{6}\left[\frac{(x-x_i)^3}{\Delta_i} - \Delta_i(x-x_i)\right] + f_i\left(\frac{x_{i+1}-x}{\Delta_i}\right) + f_{i+1}\left(\frac{x-x_i}{\Delta_i}\right)$$

$$\cancel{= \frac{(0.01892)}{6}\left[\frac{(x-2)^3}{2} - 2(x-2)\right]} -$$

$$g_2(x) = \frac{g_1''}{6}\left[\frac{(x_2-x)^3}{h} - h(x_2-x)\right]$$

$$+ \frac{g_2''}{6}\left[\frac{(x-x_1)^3}{h} - h(x-x_1)\right]$$

$$+ f_1\left(\frac{x_2-x}{h}\right) + f_2\left(\frac{x-x_1}{h}\right)$$

$$g_2(x) = \frac{\overset{=(0.01892)}{\cancel{g_1''}}}{6}\left[\frac{(\overset{4-x}{\cancel{4-x-2}})^3}{2} - 2(\overset{4-x}{\cancel{x-2}})\right]$$

$$+ \frac{\overset{=(0.34642)}{\cancel{g_2''}}}{6}\left[\frac{(x-2)^3}{h} - h(x-2)\right]$$

$$+ (-0.12828)\left(\frac{4-x}{2}\right) + (-0.10022)\left(\frac{x-2}{2}\right)$$

Similarly,
We can repeat same for other segments.

$\boxed{Q.\cancel{3}2}$

$\cancel{x}t$	0	2.0	4.0	6.0
v	0.12507	-0.12828	-0.10022	0.665

Lagrange polynomial $p(x)$

$$p(x) = (0.12507)\cdot\frac{(x-2.0)(x-4.0)(x-6.0)}{(0-2.0)(0-4.0)(0-6.0)}$$

$$+ (-0.12828)\cdot\frac{(x-0)(x-4.0)(x-6.0)}{(2-0)(2-4.0)(2-6.0)}$$

$$+ (-0.10022)\frac{(x-0)(x-2.0)(x-6.0)}{(4-0)(4-2.0)(4-6.0)}$$

$$+ (0.665)\frac{(x-0)(x-2.0)(x-4.0)}{(6.0-0)(6.0-2.0)(6.0-4.0)}$$

Q.3

Evaluating slope using Padé scheme.

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} f_0' \\ f_1' \\ f_2' \\ f_3' \\ f_4' \\ f_5' \\ f_6' \\ f_7' \end{bmatrix} = \frac{1}{2h} \begin{bmatrix} -\frac{5}{2}(f_0) + 2f_1 + \frac{1}{2}f_2 \\ 3(f_2 - f_0) \\ 3(f_3 - f_1) \\ \vdots \\ \vdots \\ 3(f_n - f_{n-2}) \\ \frac{5}{2}f_n - 2f_{n-1} - \frac{1}{2}f_{n-2} \end{bmatrix}$$

Based on given data for velocity & time
Substitute $h = 2$, f

$f_0 = 0.12507$ $\quad f_1 = -0.12828$, $\quad f_2 = -0.10022$ ... $\quad f_7 =$

$f_3 = 0.665$, $\quad f_4 = 0.665$, $\quad f_5 = -0.10022$

$f_6 = -0.12828$ $\quad f_7 = 0.12507$.

& solve to get

$$\vec{f}' = \begin{bmatrix} -0.79092 \\ 0.31729 \\ -0.24374 \\ 0.30591 \\ -0.45225 \\ 0.71164 \\ -1.20713 \\ 2.57059 \end{bmatrix} \begin{bmatrix} -0.022008 \\ -0.143832 \\ 0.259402 \\ 0.296142 \\ 0.296142 \\ -0.259402 \\ 0.143833 \\ 0.022007 \end{bmatrix}$$

Q. 4

a) Rectangular method $\neq$ Rectangle Rule = Midpoint Rule

Note $\rightarrow$ Assuming given value of the function at x_i ~~f~~ i.e. f_i to be constant for $x_i < x < x_{i+1}$ also many a times called as a rectangle rule which it is not.

As per standard text

Rectangle rule $\rightarrow$ evaluate value of function at the midpoint y_i within the given interval $x_i < x < x_{i+1}$

~~if.~~ i.e. $y_i = \frac{1}{2}(x_i + x_{i+1})$

& $f_i = f(y_i)$ is assumed constant.

Since it involves evaluation of a function f, which is unknown here, it ~~it~~ is ~~is~~ not applicable.

b) Trapezoidal $\rightarrow$

This method is easy to apply when few data points are available. It can be used here with the global accuracy $O(h^2)$

c) Simpsons method $\rightarrow$

This method is definitely better than trapezoidal rule however with a catch that it needs odd number of points & hence if number of points are even (which is the case here) it is uncertain if better accuracy will be possible(?) by approximating addition point to make it odd number of datapoints. However, more preferable.

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Q.4

(d) Gaussian Quadrature method:

It involves evaluation of a function in the interior of domain. As we have here only data points & no definition/formula of the underlying function it is NOT possible to use this method.

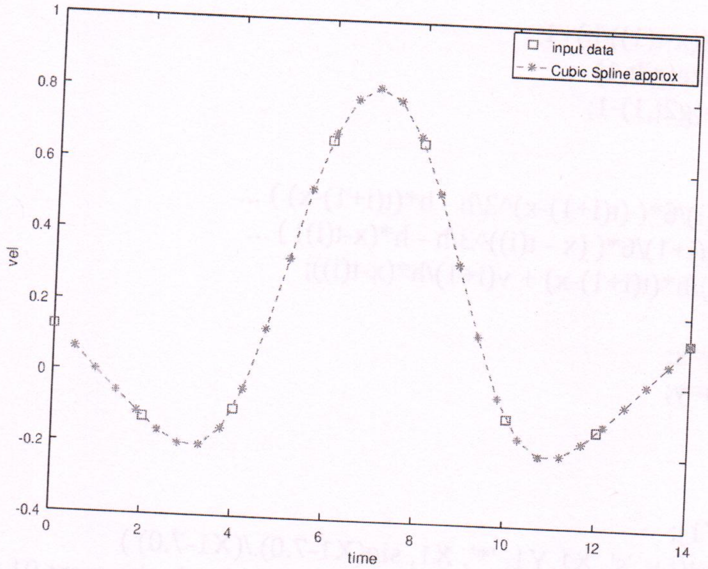
Q.5

$$I = h \cdot \left[\frac{1}{2} f_0 + \frac{1}{2} f_n + \sum_{i=1}^{n-1} f_i \right]$$

$$= 2 \left[\frac{1}{2}(0.12507) + \frac{1}{2}(0.12507) + (-0.12828) + (-0.10022) + (0.665) + (0.665) + (-0.10022) + (-0.12828) \right]$$

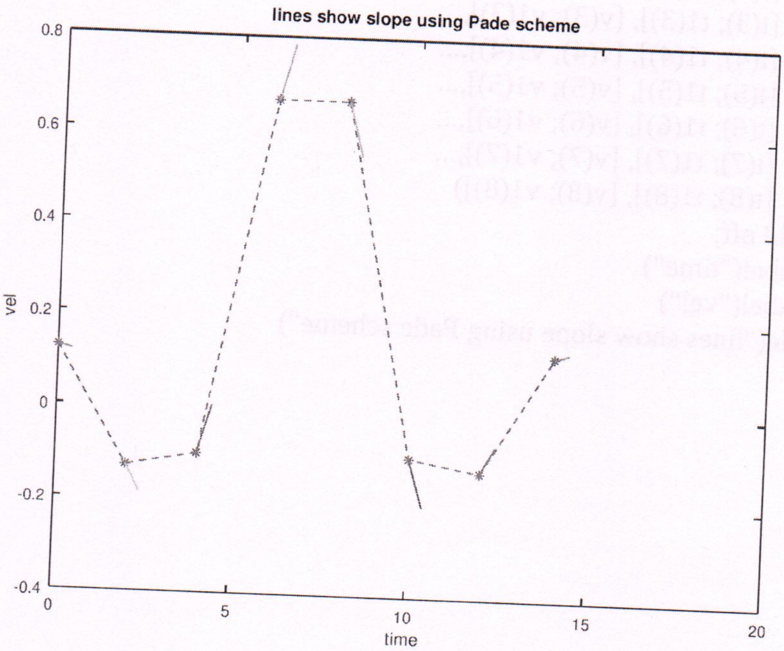
$$= 1.9961$$

```
% Engg. Maths for Adv. Studies - Autumn 2019 IIT Dharwad
% Module Numerical Methods Assignment 01
% Sample Solution using Octave - Question 1
% Author - Samarth Raut
% Last modified - 17th Nov. 2019
%****************************************************************

% Code for Cubic Spline demonstration using 8 datapoints
% dataTable=[ SrNo | x | y]

% Assumption - 8 equispaced datapoints

clear;
% Sinc function
dataTable = ...
[0  0.0     0.12507;
 1  2.0    -0.12828;
 2  4.0    -0.10022;
 3  6.0     0.66500;
 4  8.0     0.66500;
 5 10.0    -0.10022;
 6 12.0    -0.12828;
 7 14.0     0.12507]

% sin function
%dataTable = ...
%[ 1  0.00000  0.00000
%  2 0.44880  0.43388
%  3 0.89760  0.78183
%  4 1.34640  0.97493
%  5 1.79520  0.97493
%  6 2.24399  0.78183
%  7 2.69279  0.43388
%  8 3.14159  0.00000]

t=dataTable(:,2);
v=dataTable(:,3);

h=t(2)-t(1)  % Assumption - equispaced points on x axis

A=[1.0    0     0     0     0     0     0     0;
   h/6  2*h/3 h/6   0     0     0     0     0;
   0    h/6  2*h/3 h/6    0     0     0     0;
   0     0    h/6  2*h/3 h/6    0     0     0;
   0     0     0    h/6  2*h/3 h/6    0     0;
   0     0     0     0    h/6  2*h/3 h/6    0;
   0     0     0     0     0    h/6  2*h/3 h/6;
   0     0     0     0     0     0     0    1.0; ]

f=zeros(size(v,1),1);
%Free Run out conditions
```



```
f(1,1)=0;
f(size(v,1),1)=0;

for i=2:(size(v,1)-1)
  f(i)=(v(i-1)-2*v(i)+v(i+1))/h;
end
f
g2t=A\f

%Evaluation of the function value using Cubic Spline for given x
j=1;
span=t(size(t,1),1)-t(1);
nFrag=30;
for k=1:nFrag;
 x=t(1) + k/nFrag*span;

 i=floor((x-t(1))/h)+1;
 if i>=size(g2t,1)
  i=size(g2t,1)-1;
 end

 y = g2t(i)/6*( (t(i+1)-x)^3/h - h*(t(i+1)-x) ) ...
   + g2t(i+1)/6*( (x - t(i))^3/h - h*(x-t(i)) ) ...
   + v(i)/h*(t(i+1)-x) + v(i+1)/h*(x-t(i));

 X1(j,1)=x;
 Y1(j,1)=y;
 j=j+1;
end

[X1, Y1];
%%plot(t,v, 's', X1,Y1, '*', X1, sin(X1-7.0)./(X1-7.0) )
%% Analytic does not overlap  as input data for Assignment 01 was scaled in addition to
translation...Oops!:)
plot(t,v, 's', X1,Y1, '*--')
legend("input data", "Cubic Spline approx", " Analytic Sinc Fn")
%plot(t,v, 'sf', X1,Y1, '*', X1, sin(X1) )
%legend("input data", "Cubic Spline approx", " Analytic Sin Fn")
xlabel("time")
ylabel("vel")
```

```
% Engg. Maths for Adv. Studies - Autumn 2019 IIT Dharwad
% Module Numerical Methods Assignment 01
% Sample Solution using Octave - Question 3
% Author - Samarth Raut
% Last modified - 17th Nov. 2019
%**********************************************************

clear;

A=[ 1 2 0 0 0 0 0 0;
    1 4 1 0 0 0 0 0;
    0 1 4 1 0 0 0 0;
    0 0 1 4 1 0 0 0;
    0 0 0 1 4 1 0 0;
    0 0 0 0 1 4 1 0;
    0 0 0 0 0 1 4 1;
    0 0 0 0 0 0 2 1 ];

dataTable = ...
[0  0.0     0.12507;
 1  2.0    -0.12828;
 2  4.0    -0.10022;
 3  6.0     0.66500;
 4  8.0     0.66500;
 5 10.0    -0.10022;
 6 12.0    -0.12828;
 7 14.0     0.12507]

% sin function
%dataTable = ...
%[ 1  0.00000   0.00000
%  2 0.44880   0.43388
%  3 0.89760   0.78183
%  4 1.34640   0.97493
%  5 1.79520   0.97493
%  6 2.24399   0.78183
%  7 2.69279   0.43388
%  8 3.14159   0.00000]

h = dataTable(2,2)-dataTable(1,2); % assumption of equispaced points
v=dataTable(:,3);
n=size(v,1);
f = zeros( n,1);
f(1) = ((-(5/2)*v(1) + 2.0*v(2) + 0.5*v(3)))/h ;
f(n) = (((5/2)*v(n) - 2.0*v(n-1) - 0.5*v(n-2)))/h ;
for i=2:(n-1)
      f(i) = (3.0*(v(i+1)-v(i-1)))/h;
end
f
df = A\f
```



```
%Visualization
t = dataTable(:,2);
deltaT=0.4
t1= t+deltaT;
v1= v+df*deltaT;
[t,v,t1,v1]

plot(t,v, '*--');
hold on;
plot( [t(1); t1(1)], [v(1); v1(1)], ...
      [t(2); t1(2)], [v(2); v1(2)],...
      [t(3); t1(3)], [v(3); v1(3)],...
      [t(4); t1(4)], [v(4); v1(4)],...
      [t(5); t1(5)], [v(5); v1(5)],...
      [t(6); t1(6)], [v(6); v1(6)],...
      [t(7); t1(7)], [v(7); v1(7)],...
      [t(8); t1(8)], [v(8); v1(8)])
hold off;
xlabel("time")
ylabel("vel")
title("lines show slope using Pade scheme")
```