Final Exam - Module - Linear Algebra

Engineering Mathematics for Advanced Studies

IIT Dharwad

Autumn 2019

Time - 2 Hours (2:30pm -4:30pm, 28th Sept. 2019)
Maximum score - 120; Minimum score -
Rule for absentee - Minimum 30% penalty; discuss reasons absense in person to get a chance for re-test.

1. * There is 100% negative marking for most of the True/False and Yes/No questions. Marginal note has * in those cases. e.g. Incorrect answer to a question with (Marks 2*) will fetch penalty of -2 mark.
2. Worked out solutions are must for some problems.
3. Answer to question 7 is expected in the question paper itself
4. Please ensure to write Question number in a box as a heading to the upcoming answer on suppliments e.g.

Question 1

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1. Let $\boldsymbol{P}$ be a plane in $\mathbb{R}^3$containing vector elements $\boldsymbol{v}(x, y, z)$ which satisfy $3x + 2y + 2z = 3$.
   (a) What is the equation of parallel plane $\boldsymbol{P_0}$ which passes through the origin? (Marks 2*)
   (b) Is $\boldsymbol{P}$ a subspace of $\mathbb{R}^3$? (Marks 2*)
   (c) Is $\boldsymbol{P_0}$ a subspace of $\mathbb{R}^3$? (Marks 2*)

2. Please answer either Yes or No.
   Which of the following subsets of $\mathbb{R}^3$are actually subspaces:
   For the plane of vectors $\boldsymbol{b}(b_1, b_2, b_3)$
   (a) with first component $b_1 = 0$ (Marks 2*)
   (b) with first component $b_1 = 1$ (Marks 2*)
   (c) with $b_2 * b_3 = 0$ (this is the union of two subspaces: plane $b_2 = 0$ and $b_3 = 0$ (Marks 2*)
   (d) all combinations of two given vectors (1,1,0) and (2,0,1) (Marks 2*)
   (e) plane of vectors satisfying $b_3 - b_2 + 3b_1 = 0$ (Marks 2*)

3. Please answer: (Explicit evaluation of the matrix multiplications and the determinants does not carry any marks)
   (a) Verify the following set of basis vectors qualify as the basis for entire $\mathbb{R}^4$? (Marks 4)

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, v_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

   (b) Let A be a 4×4 matrix whose ***columns*** are formed by above four vectors in same sequence. True or false : at least one of the pivot value is expected to be zero. (Marks 2*)
   (c) What will be the determinant of the$D$ given by $D = (A * B^T * C)^2$ where B is obtained by swapping second and third row of the matrix A^T and C is obtained by adding 2 times 3rd row of A to 4th row of A? (Marks 4)

4. Say you have m linear algebraic equations in n variables; in matrix form we write $Ax = b$. State true or false:

(a) If $n = m$ there is always at most one solution. (Marks 2*)

(b) If $n > m$ you can always solve $Ax = b$ (Marks 2*)

(c) If $n > m$ the nullspace of A has dimension greater than zero (Marks 2*)

(d) If $n < m$ then for some b there is no solution of $Ax = b$ (Marks 2*)

(e) If $n < m$ the only solution of $Ax = 0$ is $x = 0$ (Marks 2*)

5. Let U and V both be four-dimensional subspaces of $\mathbb{R}^7$, and let $W = U \cap V$.

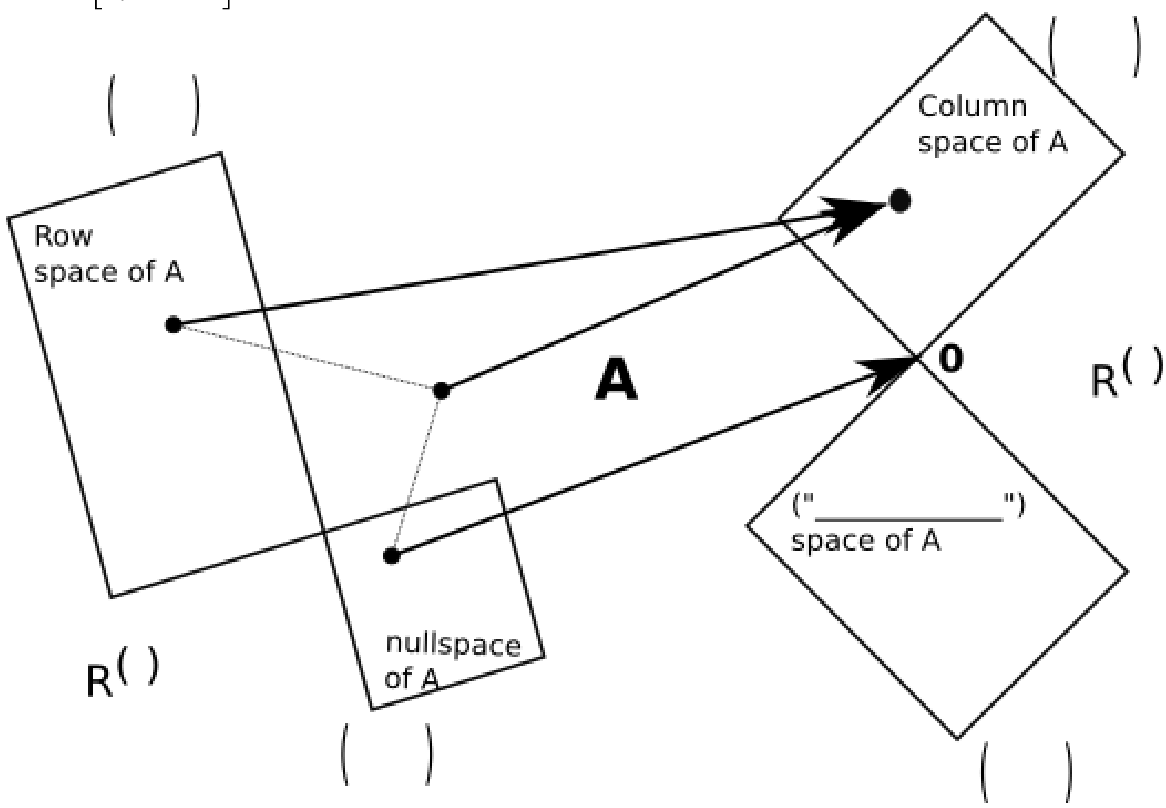
(a) Will W be always a vector space? (Yes/No) (Marks 2*)

(b) If W is a vector space, what are possible values for the dimension of W ? (Hint: Create an example to form and verify your thoughts) (Marks 2*)

6. Perform LU decomposition for matrix A given below: (Marks 3)

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

7. If $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ is a linear transformation



(a) find a vector x_1 in the null space of A (Marks 4)

(b) find a new vector x_3 by adding vector $x_2 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$ to the x_1. Find b such that $Ax_3 = b$. (Marks 1)

(Note - if you are not sure about actual x_1 calculation in above (a) assume it to be $x1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$ to proceed further)

(c) Leveraging the fact that A is a linear transformation i.e. $Ax_3 = A(x_1 + x_2)$, represent **this particular** $Ax_3 = b$ in above diagram:

i. Enter correct number in the empty brackets above R. i.e. $\mathbb{R}^{(?)}$ (two places) (Marks 1+1)

ii. name the missing "____" name of the space (one place) (Marks 1)

iii. enter 4 correct numbers depicting the dimesion of the respective 4 spaces in the following diagram (empty brackets near the rectangles) (Marks 1+1+1+1)

iv. above four vectors, x_1 , x_2 , x_3 , b at the correct locations (4 filled circles) (Marks 1+1+1+1)

8. If Rows of $A_{m\times n}$ matrix are linearly independent, then

   (a) possible value(s) for rank A = ? (Marks 1)

   (b) dimension of column space = ? (Marks 1)

   (c) dimension of left null space = ? (Marks 1)

9. At noon the minute and hour hands of a clock coincide, What in the first time,T1, when they are perpendicular? (Use concepts learnt in the class, no marks for only final answer) (Marks 10)

10. Is following transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$ a linear transformation ? (Marks 4)
(no marks awarded even for right answer if the required process/explanation is not provided. Attach worked out solution on suppliment papers.)

$$T(x_1, x_2) = \sqrt{x_1^2 + x_2^2}$$

note- vector addition operations and multiplication by scalar is defined as:

$$\left\{ \begin{matrix} x_1 \\ x_2 \end{matrix} \right\} + \left\{ \begin{matrix} x_3 \\ x_4 \end{matrix} \right\} = \left\{ \begin{matrix} x_1 + x_3 \\ x_2 + x_4 \end{matrix} \right\}$$

$$\alpha \left\{ \begin{matrix} x_1 \\ x_2 \end{matrix} \right\} = \left\{ \begin{matrix} \alpha x_1 \\ \alpha x_2 \end{matrix} \right\}$$

Answer: _________ (Yes/No)

11. Project vector b onto a line through a to get projection vector p along a. Error is $e = b - p$.

   (a) Report e and norm-2 magnitude (i.e. length) of error vector in eucleadian space. (Marks 4)
   $b = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ $a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

   (b) Please provide matrix$P$ representing Projection operation such that given any vector $y \in \mathbb{R}^3$ we can get its projection $t = Py$ (Marks 4)

12. Q is an orthogonal transformation, and $v_1 = Qu_1$ and $v_2 = Qu_2$. Given that the vectors $u_1 = \begin{bmatrix} 3 \\ 2 \\ -6 \end{bmatrix}$ and$u_2 = \begin{bmatrix} -3 \\ 4 \\ -6 \end{bmatrix}$ which all of the following are appropriate for innerproduct $v_1.v_2$: (Marks 2)

   (a) $v_1.v_2 <$ -33

   (b) $v_1.v_2 >$ 33

   (c) -66 $< v_1.v_2 <$ 66

   (d) $v_1.v_2$=0

13. If transformation A has following eigen vectors and eigen values:
$\lambda_1 = 5;\ v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\lambda_2 = 0;\ v_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

   (a) A is diagonalizable - True or False (Marks 2*)

   (b) A is invertible - True or False (Marks 2*)

   (c) What will be determinant ofA^3? (Marks 2)

(d) We can reconstruct A using above information. Can you provide explicit matrix representation for A? (Marks 4)

(e) What will be A^3? (Marks 4)

14. Verify if:

(a) following matrix Hermitian? (Marks 2)

$$A = \begin{bmatrix} 1 & 1-i \\ 1+i & 2 \end{bmatrix}$$

(b) following matrix is unitary? (Marks 2)

$$B = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$$

15. Which **all** of the following conditions imply that a matrix is semi-positive definite? (Marks 4)

(a) determinant is positive

(b) determinant is non-negative

(c) no pivot is negative

(d) $v \cdot Av \geq 0$ for all eigen vectors v of A

(e) $x \cdot Ax \geq 0$ for any vector x

16. If $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

(a) Please mention the dimensions $(m \times n)$ of the matrices and vectors in its Singular Value Decomposition (SVD) (Marks 4)

(b) For some other matrix B it is found that out of total 20 eigen values(λ_i) calculated during the SVD, $\sqrt{\lambda_1^2 + \lambda_8^2 + \lambda_{10}^2 + \lambda_{14}^2} > 0.95 * \sqrt{\sum_{i=1}^{i=20} \lambda_i^2}$. Where and how can this information be leveraged?(Only max. 3-4 lines answer is expected..but 4 marks) (Marks 4)

17. Suppose $u_1, u_2, u_3, ..., u_n$ and $v_1, v_2, v_3, ..., v_n$ are orthonormal bases for $\mathbb{R}^n$. Construct the matrix A that will transform each v_j into u_j i.e. $u_1 = Av_1, u2 = Av_2, ..., u_n = Av_n$. (Marks 4)