A generalization of Brussels sprouts

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Abstract

The Brussels sprouts game is a two-player pen and paper game which has connections with the structural properties of planar graphs. We generalize the game for all hereditary graph classes and study it for the family of forests, graphs on orientable and non-orientable surfaces of genus ≥ 0, sparse planar graphs, etc. In the process, we also introduce a new game called Circular sprouts and study it as a tool to solve problems on Brussels sprouts.

Keywords: Sprouts, impartial games, nimber, planar graphs.

1 Introduction

In 1967, Conway and Paterson [1] introduced the two player pen and paper game called *Sprouts*. The game starts with n spots (vertices) on a paper and two players place their moves alternately. A valid move consists of connecting any spot to itself or another spot with a curve (edge) and then placing a new spot on the curve drawn (subdivision). There are two restrictions that needs to be maintained during a move: the curve should not cross itself or any other curve and a spot can have at most three lines incident to it (degree three). The first player that cannot make a move, loses. The restrictions make it clear that the structure, thus obtained, remain planar throughout the game.

Since it is a finite game with no possibility for a draw, there must exist a winning strategy for either Player 1 or Player 2 based on the initial number of spots. Finding winning solutions by hand even when initiating the game with a low number of spots seem difficult, which makes it interesting to study the game. The most recent hand written analysis done was for seven spots following which only computer generated analysis has been possible [2]. It was conjectured [2] that the first player has a winning strategy if and only if the number of spots, when divided by six, leaves a remainder of three, four, or five. To date, it has been possible to verify [2] the correctness of the conjecture when the initial number of spots is $n \in \{1, 2, \cdots, 44, 46, 47, 53\}$.

Conway later introduced an extension to Sprouts, called *Brussels sprouts*, possibly as a potential way to approach the study of Sprouts. This is also a two player pen and paper game where, instead of spots, we start with n crosses. Each cross has four open arms or *open tips* and a player can only connect the open tips. So a valid move consists of connecting any two open tips with a curve and a crossbar is placed anywhere on the newly made curve. The crossbar creates two new open tips which can be used in subsequent moves. This game also retains restriction of the curves not crossing each other during the play from sprouts. As a two player game, each player plays on alternate turns and the player who cannot make a valid move loses. However, in this case, using the Euler's formula for planar graphs, one can easily figure out the player having winning strategy. In fact, the moves made by the players are redundant, and no matter how you play, the total number of moves and the winner of the game is a function of the number n of the initial crosses only.

In this article, we look into a generalised version of Brussels sprouts, where instead of crosses we have variable open tips for each spot. We also restrict the intermediate steps to certain hereditary graph classes and study them.

Given a hereditary graph family $\mathcal{F}$, We define the game *n-Brussels sprouts* for $\mathcal{F}$ with parameters $(t_1, t_2, \cdots, t_n)$, denoting it as $BS_n(\mathcal{F} : t_1, t_2, t_3, \ldots, t_n)$, as follows. The game $BS_n(\mathcal{F} : t_1, t_2, t_3, \ldots, t_n)$ starts with n spots, having $t_1, t_2, \cdots, t_n$ open tips, respectively. A valid move consists of joining two open tips with a curve followed by drawing a crossbar on the curve to create two new open tips. The graph obtained by considering the spots and intersections of a curve and a crossbar as vertices, and the curves joining two such vertices as edges, must remain inside the family $\mathcal{F}$ at all times. The first player unable to provide a valid move, loses. We will follow standard graph notations according to West [3] throughout this article unless otherwise stated.

In this article, Section 2 studies the possible number of moves and winning strategies for n-Brussels sprouts for the families of forests, and graphs on orientable and non-orientable surfaces of genus $k \geq 0$. In Section 3, we focus on the families of sparse planar graphs. In Section 4, we introduce a new, related game called *Circular sprouts*. We explore a relation between a particular class of Circular sprouts game and $BS_2(\mathcal{P} : p, q)$, where $\mathcal{P}_4$ denotes the family of triangle-free planar graphs. We figure out all nimbers for the above mentioned class of Circular sprouts, which helps us analyse the game $BS_2(\mathcal{P} : p, q)$. Finally, we conclude the article in Section 6 with the remark that the game Circular sprouts is interesting on its own merit, and maybe a potential tool for attacking the Sprouts Conjecture.

2 Forests and graphs on surfaces

To begin the study, let us first consider the family of forests.

Theorem 2.1. *Let $\mathcal{F}_t$ be the family of forests. Then $BS_n(\mathcal{F}_t : t_1, t_2, ..., t_n)$ ends after exactly $(n-1)$ moves.*

Proof. Since $\mathcal{F}_t$ is the family of forests, the resultant graph of terminated $BS_n(\mathcal{F} : t_1, t_2, ..., t_n)$ must be a tree. That is, it is possible to make a move until we create a tree. On the other hand, once we create a tree through our game, it is not possible to make any other move, as it will create a cycle.

Suppose that the game is terminated after x moves and that G is the resultant graph. Therefore, $|V(G)| = (n + x)$ as we started with n vertices and in each move we have added exactly one vertex. Also, $|E(G)| = 2x$ as we started with no edges and in each move we have added exactly two edges. We know that G is a tree, and therefore, $|E(G)| = |V(G)| - 1$. Hence we have $2x = (n + x) - 1$, which implies, $x = (n - 1)$. □

Next we will move our attention to the family $\mathcal{O}_k$ of graphs that can be drawn on orientable surfaces of genus k without crossings.

Theorem 2.2. *Let $\mathcal{O}_k$ be the family of graphs that can be drawn on orientable surfaces of genus k without crossings. Then the only possible numbers of moves until the game $BS_n(\mathcal{O}_k : t_1, t_2, ..., t_n)$ terminates are*

$$(n-2) + 2j + \sum_{i=1}^{n} t_i,$$

where $j = 0, 1, \cdots, k$.

Proof. Suppose the game ends after x moves and the resultant graph after the end of the game be G. Thus, $|V(G)| = n + x$ and $|E(G)| = 2x$ as we start with n vertices, 0 edges, and include exactly one vertex and two edge in each move.

Furthermore, we observe that the game cannot end if a particular face contains two or more open tips, while in the last move involved in creating a particular face of G will ensure at least one open tip inside the face. Thus, the number of open tips is equal to the number of faces of G, that is, $|F(G)| = \sum_{i=1}^{n} t_i$.

Notice that, even though G is embedded on $\mathcal{O}_k$, it maybe possible to embed it on an orientable surface of genus less than k. Let j be the least number for which G can be embedded on $\mathcal{O}_j$. Thus, G will satisfy the Euler's formula

$$|V(G)| - |E(G)| + |F(G)| = 2 - 2j$$

for orienatable surfaces.

Thus, by replacing the values of $|V(G)|$, $|E(G)|$ and $|F(G)|$ we get

$$x + n - 2x + \sum_{i=1}^{n} t_i = 2 - 2j \implies x = (n-2) + 2j + \sum_{i=1}^{n} t_i$$

which completes the proof. □

Recall that the orientable surface with genus 0 is nothing but the sphere, and thus, the family $\mathcal{O}_0$ of graphs are nothing but planar graphs. The above theorem characterizes all possible number of moves for the game $BS_n(\mathcal{O}_k : t_1, t_2, \cdots, t_n)$ to end. Clearly, when $k = 0$, that is, for the game $BS_n(\mathcal{O}_0 : t_1, t_2, \cdots, t_n)$ the game will end after exactly $(n-2) + \sum_{i=1}^{n} t_i$ moves. Therefore, for planar graphs, the game will end after a constant number of moves, and the first player will win if and only if that constant is odd irrespective of how the game gets played.

Corollary 2.3. *The game $BS_n(\mathcal{O}_0 : t_1, t_2, \cdots, t_n)$ will end exactly after $(n-2) + \sum_{i=1}^{n} t_i$ moves and the first player will win if and only if $n + \sum_{i=1}^{n} t_i$ is odd.*

Proof. Follows directly from Theorem 2.2 by restricting it for $k = 0$. □

On the other hand, if we consider the game $BS_n(\mathcal{O}_k : t_1, t_2, \cdots, t_n)$ for all $k \geq 1$, even though the number of moves after which the game may end is not a constant, note that it only differs by an even number. Thus we have the following.

Corollary 2.4. *In the game $BS_n(\mathcal{O}_k : t_1, t_2, \cdots, t_n)$ for $k \geq 1$, the first player will win if and only if $n + \sum_{i=1}^{n} t_i$ is odd.*

Proof. Follows directly from Theorem 2.2 by observing that $(n-2) + 2j + \sum_{i=1}^{n} t_i$ is odd if and only if $n + \sum_{i=1}^{n} t_i$ is odd for all $j = 0, 1, \cdots, k$. □

On a similar vein, we also study the family $\mathcal{N}_k$ of graphs that can be drawn on non-orientable surfaces of genus k without crossings.

Theorem 2.5. *Let $\mathcal{N}_k$ be the family of graphs that can be drawn on non-orientable surfaces of genus k without crossings. Then the only possible numbers of moves until the game $BS_n(\mathcal{N}_k : t_1, t_2, ..., t_n)$ terminates are*

$$(n-2) + j + \sum_{i=1}^{n} t_i,$$

where $j = 0, 1, \cdots, k$.

Proof. Suppose the game ends after x moves and the resultant graph after the end of the game be G. Thus, $|V(G)| = n + x$ and $|E(G)| = 2x$ as we start with n vertices, 0 edges, and include exactly one vertex and two edge in each move.

Furthermore, we observe that the game cannot end if a particular face contains two or more open tips, while in the last move involved in creating a particular face of G will ensure at least one open tip inside the face. Thus, the number of open tips is equal to the number of faces of G, that is, $|F(G)| = \sum_{i=1}^{n} t_i$.

Notice that, even though G is embedded on $\mathcal{N}_k$, it maybe possible to embed it on an non-orientable surface of genus less than k. Let j be least number for which G can be embedded on $\mathcal{N}_j$. Thus, G will satisfy the Euler's formula

$$|V(G)| - |E(G)| + |F(G)| = 2 - j$$

for non-orienatable surfaces.

Thus, by replacing the values of $|V(G)|$, $|E(G)|$ and $|F(G)|$ we get

$$x + n - 2x + \sum_{i=1}^{n} t_i = 2 - j \implies x = (n-2) + j + \sum_{i=1}^{n} t_i$$

which completes the proof. □

Recall that the non-orientable surface with genus 0 is nothing but the projective plane, and thus, the family $\mathcal{N}_0$ of graphs are nothing but the projective planar graphs. Therefore the following corollary follows directly.

Corollary 2.6. *The game $BS_n(\mathcal{N}_0 : t_1, t_2, \cdots, t_n)$ will end exactly after $(n-2) + \sum_{i=1}^{n} t_i$ moves and the first player will win if and only if $n + \sum_{i=1}^{n} t_i$ is odd.*

Proof. Follows directly from Theorem 2.5 by restricting it for $k = 0$. □

However, in this case, for higher genus, unlike in the case of orientable surfaces, the parity of the number of moves after which the game may end is not the same. Therefore, depending on how the game is played, it maybe won by Player 1 or Player 2. Thus, it makes sense to analyse winning strategy. We pose this as an open question.

Question 2.7. *Which player has an winning strategy for the game $BS_n(\mathcal{N}_k : t_1, t_2, \cdots, t_n)$ when $k \geq 1$?*

3 Sparse planar graphs

In this section, let us focus on the family $\mathcal{P}_g$ of planar graphs with girth at least g. The first result shows that if we fix a particular value of n, then number of moves after which the game $BS_n(\mathcal{P}_g : t_1, t_2, ..., t_n)$ end is a constant for large values of g.

Theorem 3.1. *Let $\mathcal{P}_g$ be the family of planar graphs having girth at least g. Then the game $BS_n(\mathcal{P}_g : t_1, t_2, \cdots, t_n)$ game ends exactly after $(n-1)$ moves for all $g \geq 2n+1$.*

Proof. Let G_x be the resultant graph after x number of moves. As we start with n vertices and add one vertex in each move, we have $|V(G_x)| = (n+x)$. Also, we start with zero edges and add two edges in each move. Thus we have $|E(G_x)| = 2x$.

Thus after n moves, we have $2n$ vertices and $2n$ edges in G_n. This graph must have a cycle, and as the graph has only $2n$ vertices, the cycle cannot have length greater than or equal to $(2n+1)$. This is a contradiction.

Hence, the number of moves cannot be more than $(n-1)$. On the other hand, $(n-1)$ moves are also minimum number of moves due to Theorem 2.1. □

Next, we focus particularly on the family of triangle-free planar graphs, that is, $\mathcal{P}_4$. We find upper and lower bounds of the number of moves after which the game $BS_n(\mathcal{P}_4 : t_1, t_2, ..., t_n)$ ends.

Theorem 3.2. *The number of moves after which the game $BS_n(\mathcal{P}_4 : t_1, t_2, ..., t_n)$ ends is between $(4+n)$ and $(n-2) + \sum_{i=1}^{n} t_i$, where $n \geq 2$ and $t_i \geq 3$.*

Proof. Suppose the game ends after x moves and the resultant graph after the end of the game be G which is a planar graph, in particular. Thus, due to Corollary 2.3 we have $x \leq (n-2) + \sum_{i=1}^{n} t_i$.

As we start with n vertices and add one vertex in each move, we have $|V(G_x)| = (n+x)$. Also, as we start with zero edges and add two edges in each move, we have $|E(G_x)| = 2x$.

Furthermore, since $t_i \geq 3$ and $n \geq 2$, we are forced to have $|F(G)| \geq 6$. Hence by Euler's Formula we have

$$\begin{aligned} |V(G)| - |E(G)| + |F(G)| = 2 &\implies (x+n) - 2x + 6 \leq 2 \\ &\implies x \geq 4 + n \end{aligned}$$

Therefore, $(4+n) \leq x \leq (n-2) + \sum_{i=1}^{n} t_i$. □

A natural question we can ask here is whether there is a play of $BS_n(\mathcal{P}_4 : t_1, t_2, ..., t_n)$ that ends after $(n-2) + \sum_{i=1}^{n} t_i$ moves and one that ends after $(4+n)$ moves. In the next result, we will see that indeed for $n = 2$, such plays exist when the ratio of p and q are at most two.

Theorem 3.3. *There exists plays of $BS_2(\mathcal{P}_4 : p, q)$ which ends after $(p+q)$ and 6 moves, respectively, for $p \leq q \leq 2p$.*

Proof. Let the two spots (vertices) present in the initial stage of the game $BS_2(\mathcal{P}_4 : p, q)$ be x and y positioned on a horizontal line, x being in the left side of y. Furthermore, let $x_1, x_2, ..., x_p$ be the open tips coming out of x, arranged in a clockwise order around x and let $y_1, y_2, ..., y_q$ be the open tips coming out of y, arranged in an anti-clockwise order around y.

First, we are going to describe the play of $BS_2(\mathcal{P}_4 : p, q)$ which ends after $(p+q)$ moves for $q = (p+r)$ where $0 \leq r \leq p$. Observe that it is enough to describe the required sequences of the $(p+q)$ moves in the play.

Let the first move be connecting x_1 to y_1 with a curve and putting the crossbar t_1 on it. In the subsequent moves, we connect x_i to y_{2i-1} with a curve and put the crossbar t_i on it for $i = 2, 3, \cdots, r+1$. After that we connect x_j to y_{r+j} with a curve and put the crossbar t_j for $j = r+2, r+3, \cdots, p$. That means, we have made a total of p moves till now.

Next, we connect t_i to t_{i+1} with a curve and put a crossbar s_i on it for $i = 1, 2, \cdots, p-1$, which are $(p-1)$ more moves. Then we connect s_i with y_{2i} with a curve and put a crossbar on it for $i = 1, 2, \cdots, r$, which amounts to r moves. Finally, we connect t_1 with t_p with a curve and put a crossbar on it. Observe that, this ends the play as no more moves can be made and a total of $p + (p-1) + r + 1 = 2p + r = p + q$ moves are made. Thus we are done with the first part of the proof.

Secondly, we are going to describe the play of $BS_2(\mathcal{P}_4 : p, q)$ which ends after 6 moves. Observe that it is enough to describe the required sequences of the 6 moves in the play.

The first two moves in this case are connecting x_1 to y_1 and x_2 to y_2 with two curves and putting the crossbars t_1 and t_2 on them, respectively. Notice that the above two curves divide the plane into two regions: R_1 containing the open tip x_3, R_2 not containing it. We connect t_1 and t_2 with a curve through R_1 and put a crossbar t_3 on it. Next, we connect x_p and y_3 to t_3 two curves and put crossbars on them. Finally, connect t_1 and t_2 with a curve through R_2 and put a crossbar on it. Observe that, this ends the play as no more moves can be made and a total of 6 moves are made. Thus, we are done with the second part of the proof as well. □

From the above theorem, we can notice that the game $BS_2(\mathcal{P}_4 : p, q)$ does not have a clear winner, and depending on the play, either Player 1 or Player 2 can win. Therefore, studying which player has winning strategy makes sense. However, when we tried to do it by hand, it became extremely difficult, even for small values of p and q. Hence, we felt the need for an alternative technique to attack this problem, with the hope of building a potential general technique to attack such problems. We are going to discuss this technique in the next section, while Section 5 will also contain a proof of the following.

Theorem 3.4. *There exists a winning strategy for Player 2 in the game $BS_2(\mathcal{P}_4 : p, q)$ for all $p, q \geq 3$.*

4 The Circular sprouts game

While studying the game $BS_2(\mathcal{P}_4 : p, q)$, we encountered another similar game which we found to be interesting. Let us define this new game independently, and in a generalized form, even though in this article we will study only a specific restriction of it which will help us in proving Theorem 3.4.

This new game, named the *n-Circular sprouts* game for the family $\mathcal{F}$ with parameters $(t_1, t_2, \cdots, t_n)$, is denoted by the notation $CS_n(\mathcal{F} : t_1, t_2, \cdots, t_n)$, The initial set up of this game consists of n spots $v_1, v_2, \cdots, v_n$ arranged in a clockwise order on the perimeter of a circle with v_i having t_i open tips coming out in the interior of the circle. The rest of the rules of the game is the same as Brussels sprouts with the following added constraint: the curves drawn by the players must be entirely contained in the interior of the circle.

Next let us observe how this game is related to $BS_2(\mathcal{P}_4 : p, q)$. Let the open tips around the first spot be $x_1, x_2, \cdots, x_p$ arranged in a clockwise order, and let the open tips around the second spot be $y_1, y_2, \cdots, y_q$ arranged in an anti-clockwise order. Observe that, the very first move by Player 1 in the game $BS_2(\mathcal{P}_4 : p, q)$ is unique up to renaming of the open tips. Therefore, without loss of generality one can assume that the very first move is joining the open tip x_1 with the open tip y_1 with a curve and then placing a crossbar on it. After that, the second player is forced to join an open tip x_i to an open tip y_j, for some $i, j \neq 1$. This move will enable us to write the present game as the sum of two Circular sprouts game: $CS_4(\mathcal{P}_4 : i-2, 1, j-2, 1)$ and $CS_4(\mathcal{P}_4 : p-i, 1, q-j, 1)$ for some $i \in \{2, 3, \cdots, p\}$ and $j \in \{2, 3, \cdots, q\}$. Hence it will be enough to study and understand the games of the type $CS_4(\mathcal{P}_4 : p, 1, q, 1)$.

Before moving forward with the study of these games, we would to point out that all the games discussed here are two player finite impartial games, and thus their *nimber* can be calculated. To conclude which player has a winning strategy for a particular two player impartial game, it is enough to calculate the nimber value of the game; the second player has an winning strategy if and only if the nimber of a game is 0 [4].

Recall that, to calculate the nimber of a game X, one first need to generate the entire game tree having X as its root. Next the leaves of the tree are all assigned nimber equal to 0, while for the other nodes its nimber is the least non-negative integer which does not occur as a nimber of any of its children. Let us denote the nimber of a game X by $\eta[X]$ for convenience. We know that, if an impartial game X can be written as a sum of two impartial games, Y and Z, then the nimber of X can be given by $\eta[X] = \eta[Y] \oplus \eta[Z]$, where $\oplus$ denote the XOR operation [4].

Thus our objective now is to calculate nimber of the game $CS_4(\mathcal{P}_4 : p, 1, q, 1)$ for $p, q \geq 0$. Note that, the games $CS_4(\mathcal{P}_4 : p, 1, q, 1)$ and $CS_4(\mathcal{P}_4 : q, 1, p, 1)$ are the exact same games up to symmetry. Thus it is enough to calculate $\eta[CS_4(\mathcal{P}_4 : p, 1, q, 1)]$ for all $q \geq p \geq 0$.

We assume $CS_4\mathcal{P}_4 : (p, 1, q, 1)$ to be embedded as p on the left, q on the right and the singular open tips to be at the top and bottom.

Theorem 4.1. *For all* $0 \leq p \leq q$*, we have*

$$\eta[CS_4(\mathcal{P}_4 : p, 1, q, 1)] = \begin{cases} 1 & \textit{if } p = q, \\ \frac{4}{5}(p + q - i) + 2\lfloor \frac{i}{4} \rfloor & \textit{if } p < q < 2p - \lfloor \frac{|p-2|}{2} \rfloor \textit{ where } i \equiv p + q (mod\ 5) \\ 2p & \textit{if } q \geq 2p - \lfloor \frac{|p-2|}{2} \rfloor. \end{cases}$$

where $1, 2, 3, 4, 5$ *are the representative of the integers modulo* 5.

As the proof is lengthy and complicated, we will move it to the next section.

5 Proof of Theorem 4.1 and 3.4

Let us assume that a typical pictorial representation of the game $CS_4(\mathcal{P}_4 : p, 1, q, 1)$ be as follows. Let the reference circle of the game be the circle having unit radius and $(0, 0)$ as its center. After that, let the four spots on it, along with their positions, be x at $(-1, 0)$, t at $(0, 1)$, y at $(1, 0)$ and b at $(0, -1)$. Furthermore, let $x_1, x_2, ..., x_p$ be the open tips coming out of x, arranged in a clockwise order around x and let $y_1, y_2, ..., y_q$ be the open tips coming out of y, arranged in an anti-clockwise order around y. Also let t_1 and b_1 be the open tips coming out of t and b, respectively.

Notice that, there can be two types of moves that the first player can perform on the given initial stage of the game $CS_4(\mathcal{P}_4 : p, 1, q, 1)$. The first type is to connect t_1 with b_1 with a curve and then put a crossbar on it. However, this creates a game equivalent to the sum of the two games $CS_4(\mathcal{P}_4 : p, 0, 1, 0)$ and $CS_4(\mathcal{P}_4 : 1, 0, q, 0)$. The second type of move is to join $x_{p'+1}$ with $y_{q'+1}$ with a curve and then put a crossbar on it. This creates a game equivalent to the sum of the two games $CS_4(\mathcal{P}_4 : p', 1, q', 1)$ and $CS_4(\mathcal{P}_4 : p'', 1, q'', 1)$, where $p' \in \{0, 1, \cdots, p' - 1\}$, $q' \in \{0, 1, \cdots, q' - 1\}$, $p'' = p - p' - 1$ and $q'' = q - q' - 1$. From now on, we will assume $p' \in \{0, 1, \cdots, p' - 1\}$, $q' \in \{0, 1, \cdots, q' - 1\}$, $p'' = p - p' - 1$ and $q'' = q - q' - 1$ as general convention for this proof.

Observe that, irrespective of the moves, the game becomes a sum of similar kinds of games allowing us to use induction. That is why, we will use the method of strong induction on p to prove this result. Before going to the base case, let us first prove some useful lemmas.

Lemma 5.1. *We have*

(i) $\eta[CS_4(\mathcal{P}_4 : 0, 0, q, 0)] = 0$,

(ii) $\eta[CS_4(\mathcal{P}_4 : 0, 1, 0, 1)] = 1$,

(iii) $\eta[CS_4(\mathcal{P}_4 : 1, 0, q, 0)] = 1$.

(iv) $\eta[CS_4(\mathcal{P}_4 : 0, 1, q, 1)] = 0$.

Proof. Notice that the game $CS_4(\mathcal{P}_4 : 0, 0, q, 0)$ does not have any move, and thus trivially,

$$\eta[CS_4(\mathcal{P}_4 : 0, 0, q, 0)] = 0. \tag{1}$$

Similarly, the game $CS_4(\mathcal{P}_4 : 0, 1, 0, 1)$ has exactly one move, that is, connecting t_1 and b_1 with a curve, and thus trivially,

$$\eta[CS_4(\mathcal{P}_4 : 0, 1, 0, 1)] = 1. \tag{2}$$

In the game $CS_4(\mathcal{P}_4 : 0, 1, q, 1)$ for $q \geq 1$, the only possible first move is to connect t_1 with b_1 by a curve. This can be expressed as a sum of the games $CS_4(\mathcal{P}_4 : 0, 0, 1, 0)$ and $CS_4(\mathcal{P}_4 : 1, 0, q, 0)$. We already know that $\eta[CS_4(\mathcal{P}_4 : 0, 0, q, 0)] = 0$ by equation (1), and one can observe that the only possible move in the game $CS_4(\mathcal{P}_4 : 1, 0, q, 0)$ is to connect x_1 to some $y_{q'+1}$ after which the game will end. Therefore,

$$\eta[CS_4(\mathcal{P}_4 : 1, 0, q, 0)] = 1. \tag{3}$$

Hence the only child of the game $CS_4(\mathcal{P}_4 : 0, 1, q, 1)$ has nimber

$$\eta[CS_4(\mathcal{P}_4 : 0, 0, q, 0)] \oplus \eta[CS_4(\mathcal{P}_4 : 1, 0, q, 0)] = 1 \tag{4}$$

which implies

$$\eta[CS_4(\mathcal{P}_4 : 0, 1, q, 1)] = 0. \tag{5}$$

This completes the proof. □

This implies that the formula given in the statement of Theorem 4.1 hold when $p = 0$.

Lemma 5.2. *We have*

$$\eta[CS_4(\mathcal{P}_4 : 1, 1, q, 1)] = \begin{cases} 1 & \text{if } q = 1, \\ 2 & \text{if } q \geq 2. \end{cases} \tag{6}$$

Proof. The game $CS_4(\mathcal{P}_4 : 1, 1, q, 1)$ has two types of child in the game tree. The first, obtained by connecting t_1 and b_1 with a curve, can be expressed as the sum of $CS_4(\mathcal{P}_4 : 1, 0, 1, 0)$ and $CS_4(\mathcal{P}_4 : 1, 0, q, 0)$. This game has nimber

$$\eta[CS_4(\mathcal{P}_4 : 1, 0, 1, 0)] \oplus \eta[CS_4(\mathcal{P}_4 : 1, 0, q, 0)] = 1 \oplus 1 = 0 \tag{7}$$

due to equation (2) and (3). The second child is obtained by connecting x_1 and $y_{q'+1}$, and thus can be expressed as a sum of $CS_4(\mathcal{P}_4 : 0, 1, q', 1)$ and $CS_4(\mathcal{P}_4 : 0, 1, q'', 1)$, and thus has nimber

$$\eta[CS_4(\mathcal{P}_4 : 0, 1, q', 1)] \oplus \eta[CS_4(\mathcal{P}_4 : 0, 1, q'', 1)] = 1 \oplus 1 = 0 \tag{8}$$

for $q', q'' \geq 1$ due to equation (5). On the other hand, using equation (2) and (5), when $q' = 0$ (or $q'' = 0$) we have

$$\eta[CS_4(\mathcal{P}_4 : 0, 1, 0, 1)] \oplus \eta[CS_4(\mathcal{P}_4 : 0, 1, q-1, 1)] = 1 \oplus 0 = 1 \tag{9}$$

for $q \geq 2$. Notice that, if $q = 1$, then

$$\eta[CS_4(\mathcal{P}_4 : 0, 1, 0, 1)] \oplus \eta[CS_4(\mathcal{P}_4 : 0, 1, 0, 1)] = 1 \oplus 1 = 0. \tag{10}$$

That means, the game $CS_4(\mathcal{P}_4 : 1, 1, q, 1)$ has child with nimber 0 and 1 when $q \geq 2$. On the other hand, if $q = 1$, it has child with only nimber 0. Therefore, we have

$$\eta[CS_4(\mathcal{P}_4 : 1, 1, q, 1)] = \begin{cases} 1 & \text{if } q = 1, \\ 2 & \text{if } q \geq 2. \end{cases} \tag{11}$$

This completes the proof. □

Thus, the formula given in the statement is true for $p = 1$ as well.

Hence we have verified the formula to be true for $p \leq 1$, and this is the base case of our induction. Let us assume that the formula is true for all $\eta[CS_4(\mathcal{P}_4 : p - i, 1, q, 1)]$, where $i \in \{1, 2, \cdots, p\}$. For the induction step, we need to show that it is true for $\eta[CS_4(\mathcal{P}_4 : p, 1, q, 1)]$ as well. We will break our induction step of the proof across several lemmas.

However, before that we will make a general observation useful for all cases.

Lemma 5.3. *For all* $0 \leq p \leq q$,

$$\eta[CS_4(\mathcal{P}_4 : p, 1, q, 1)] \geq 1. \tag{12}$$

Proof. Notice that each of the games $CS_4(\mathcal{P}_4 : p, 1, q, 1)$ has one child which is expressed as a sum of $CS_4(\mathcal{P}_4 : p, 0, 1, 0)$ and $CS_4(\mathcal{P}_4 : 1, 0, q, 0)$. By equation 3, this game has nimber

$$\eta[CS_4(\mathcal{P}_4 : p, 0, 1, 0)] \oplus \eta[CS_4(\mathcal{P}_4 : 1, 0, q, 0)] = 1 \oplus 1 = 0. \tag{13}$$

Therefore, all the games of the type $CS_4(\mathcal{P}_4 : p, 1, q, 1)$ has a child with nimber 0. This implies

$$\eta[CS_4(\mathcal{P}_4 : p, 1, q, 1)] \geq 1. \tag{14}$$

Hence the proof. □

Now we are ready to handle the case where $p = q$.

Lemma 5.4. *If* $\eta[CS_4(\mathcal{P}_4 : n, 1, q, 1)]$ *satisfy the formula given in the statement of Theorem 4.1 for all* $n \leq p - 1$*, then* $\eta[CS_4(\mathcal{P}_4 : p, 1, p, 1)] = 1$.

Proof. When $p = q$, note that because of Lemma 5.3 it is enough to show that none of the children of $CS_4(\mathcal{P}_4 : p, 1, q, 1)$ in the game tree has nimber equal to 1. That is, we need to show that

$$\eta[CS_4(\mathcal{P}_4 : p', 1, q', 1)] \oplus \eta[CS_4(\mathcal{P}_4 : p'', 1, q'', 1)] \neq 1.$$

Notice that, by our induction hypothesis, both $\eta[CS_4(\mathcal{P}_4 : p', 1, q', 1)]$ and $\eta[CS_4(\mathcal{P}_4 : p'', 1, q'', 1)]$ has even values unless $p' = q'$, and hence, as $\oplus$ of two even numbers is even, we are done unless $p' = q'$. If $p' = q'$, then both $\eta[CS_4(\mathcal{P}_4 : p', 1, q', 1)]$ and $\eta[CS_4(\mathcal{P}_4 : p'', 1, q'', 1)]$ are equal to 1. Thus, in this case also,

$$\eta[CS_4(\mathcal{P}_4 : p', 1, p', 1)] \oplus \eta[CS_4(\mathcal{P}_4 : p'', 1, p'', 1)] = 1 \oplus 1 = 0.$$

This implies that

$$\eta[CS_4(\mathcal{P}_4 : p, 1, p, 1)] = 1$$

and concludes the proof. □

Lemma 5.5. *If $\eta[CS_4(\mathcal{P}_4 : n, 1, q, 1)]$ satisfy the formula given in the statement of Theorem 4.1 for all $n \leq p-1$, then $\eta[CS_4(\mathcal{P}_4 : p, 1, q, 1)] = 2p$ when $q \geq 2p - \lfloor \frac{p-2}{2} \rfloor$.*

Proof. When $q \geq 2p - \lfloor \frac{p-2}{2} \rfloor$, we need to show that none of the children of $CS_4(\mathcal{P}_4 : p, 1, q, 1)$ in the game tree has nimber equal to $2p$, where as, for each $i \in \{1, 2, \cdots, 2p-1\}$, there exists a child of $CS_4(\mathcal{P}_4 : p, 1, q, 1)$ in the game tree which has nimber equal to i.

First we will show that the odd numbers less than $2p$ appear as the nimbers of the children of $CS_4(\mathcal{P}_4 : p, 1, q, 1)$ in the game tree. For that, let us consider the children that can be expressed as sum of the games $CS_4(\mathcal{P}_4 : p', 1, q', 1)$ and $CS_4(\mathcal{P}_4 : p'', 1, q'', 1)$, where $q \geq 2p - \lfloor \frac{p-2}{2} \rfloor$ and $p' = q'$. In this scenario, note that we have

$$\eta[CS_4(\mathcal{P}_4 : p', 1, q', 1)] = \eta[CS_4(\mathcal{P}_4 : p', 1, p', 1)] = 1$$

by induction hypothesis. Moreover, observe that

$$\begin{aligned} q'' = q - 1 - q' &\geq (2p - \lfloor \frac{p-2}{2} \rfloor) - 1 - p' \\ &\geq 2p - 2p' - \lfloor \frac{p''-2}{2} \rfloor + \lfloor \frac{p''-2}{2} \rfloor - \lfloor \frac{p-2}{2} \rfloor - 2 \\ &\geq 2(p - p' - 1) - \lfloor \frac{p''-2}{2} \rfloor - \left(\lfloor \frac{p-2}{2} \rfloor - \lfloor \frac{p''-2}{2} \rfloor \right) \\ &\geq 2p'' - \lfloor \frac{p''-2}{2} \rfloor. \end{aligned}$$

This implies

$$\eta[CS_4(\mathcal{P}_4 : p'', 1, q'', 1)] = 2p''$$

due to the induction hypothesis. Therefore,

$$\eta[CS_4(\mathcal{P}_4 : p', 1, p', 1)] \oplus \eta[CS_4(\mathcal{P}_4 : p'', 1, q'', 1)] = 1 \oplus 2p'' = 2p'' + 1$$

is nimber of a child of $CS_4(\mathcal{P}_4 : p, 1, q, 1)$ in the game tree. As $p'' \in \{0, 1, \cdots, p-1\}$, the above covers all odd numbers less than $2p$.

Next we will show that the even numbers less than $2p$ appear as the nimbers of the children of $CS_4(\mathcal{P}_4 : p, 1, q, 1)$ in the game tree. For that, let us consider the children that can be expressed as sum of the games $CS_4(\mathcal{P}_4 : p', 1, q-1, 1)$ and $CS_4(\mathcal{P}_4 : p'', 1, 0, 1)$. In this scenario, note that we have

$$\eta[CS_4(\mathcal{P}_4 : p'', 1, 0, 1)] = 0$$

by equation (5). Also

$$\begin{aligned} q-1 &\geq (2p - \lfloor \frac{p-2}{2} \rfloor) - 1 \\ &\geq (2p' - \lfloor \frac{p'-2}{2} \rfloor) - \left(\lfloor \frac{p-2}{2} \rfloor - \lfloor \frac{p'-2}{2} \rfloor \right) \\ &\geq 2p' - \lfloor \frac{p'-2}{2} \rfloor. \end{aligned}$$

This implies

$$\eta[CS_4(\mathcal{P}_4 : p', 1, q-1, 1)] = 2p'$$

due to the induction hypothesis. Therefore,

$$\eta[CS_4(\mathcal{P}_4 : p', 1, q-1, 1)] \oplus \eta[CS_4(\mathcal{P}_4 : p'', 1, 0, 1)] = 2p' \oplus 0 = 2p'$$

is nimber of a child of $CS_4(\mathcal{P}_4 : p, 1, q, 1)$ in the game tree. As $p' \in \{0, 1, \cdots, p-1\}$, the above covers all even numbers less than $2p$.

Now let us show that there is no child of $CS_4(\mathcal{P}_4 : p, 1, q, 1)$ in the game tree whose nimber equals $2p$. Notice that, in general

$$\eta[CS_4(\mathcal{P}_4 : p', 1, q', 1)] \leq 2p'$$

due to the induction hypothesis. Therefore,

$$\eta[CS_4(\mathcal{P}_4 : p', 1, q', 1)] \oplus \eta[CS_4(\mathcal{P}_4 : p'', 1, q'', 1)] \leq 2p' + 2p'' = 2(p' + p - p' - 1) < 2p.$$

Hence we are done. □

When $p < q < 2p - \lfloor \frac{p-2}{2} \rfloor$, proving the induction step needs a careful treatment. Due to equation (14), we know that

$$\eta[CS_4(\mathcal{P}_4 : p, 1, q, 1)] \geq 1.$$

Now to prove this case, first we need to show that

$$\eta[CS_4(\mathcal{P}_4 : p', 1, q', 1)] \oplus \eta[CS_4(\mathcal{P}_4 : p'', 1, q'', 1)] \neq \eta[CS_4(\mathcal{P}_4 : p, 1, q, 1)].$$

Lemma 5.6. *If $\eta[CS_4(\mathcal{P}_4 : n, 1, q, 1)]$ satisfy the formula given in the statement of Theorem 4.1 for all $n \leq p-1$, then*

$$\eta[CS_4(\mathcal{P}_4 : p', 1, q', 1)] \oplus \eta[CS_4(\mathcal{P}_4 : p'', 1, q'', 1)] \neq \eta[CS_4(\mathcal{P}_4 : p, 1, q, 1)]$$

where $p < q < 2p - \lfloor \frac{p-2}{2} \rfloor$.

Proof. Without loss of generality, let us assume $q' \geq p'$. When $p' = q'$, we have

$$\eta[CS_4(\mathcal{P}_4 : p', 1, q', 1)] \oplus \eta[CS_4(\mathcal{P}_4 : p'', 1, q'', 1)] \neq \eta[CS_4(\mathcal{P}_4 : p, 1, q, 1)]$$

as the left hand side is of the form $1 \oplus$ an even number, that is, an odd number while the right hand side is an even number due to induction hypothesis.

When $p' < q'$, suppose that $p' < q' < 2p' - \lfloor \frac{p'-2}{2} \rfloor$ and $p'' < q'' < 2p'' - \lfloor \frac{p''-2}{2} \rfloor$ (or $q'' < p'' < 2q'' - \lfloor \frac{q''-2}{2} \rfloor$). Moreover, let $i \equiv (p+q) \pmod 5$, $i' \equiv (p'+q') \pmod 5$, and $i'' \equiv (p''+q'') \pmod 5$. In this scenario, i' and i'' can have some specific values depending on i. If we know these values, then it is possible to compute and compare the terms

$$\eta[CS_4(\mathcal{P}_4 : p', 1, q', 1)] \oplus \eta[CS_4(\mathcal{P}_4 : p'', 1, q'', 1)].$$

and

$$\eta[CS_4(\mathcal{P}_4 : p, 1, q, 1)].$$

We present each of these cases in the following table which shows that

$$\eta[CS_4(\mathcal{P}_4 : p', 1, q', 1)] \oplus \eta[CS_4(\mathcal{P}_4 : p'', 1, q'', 1)] < \eta[CS_4(\mathcal{P}_4 : p, 1, q, 1)]$$

in this scenario. For convenience, we will demonstrate a sample calculation.

If $i = 4$, the possible values for $\{i', i''\}$ are $\{5, 2\}$, $\{1, 1\}$, and $\{3, 4\}$. When $\{i', i''\} = \{5, 2\}$

$$\begin{aligned}
\eta[CS_4(\mathcal{P}_4 : p', 1, q', 1)] \oplus \eta[CS_4(\mathcal{P}_4 : p'', 1, q'', 1)] &= \frac{4}{5}(p + q - 2 - 5 - 2) + 2 \\
&= \frac{4}{5}(p + q + 1) - 6 \\
&\neq \frac{4}{5}(p + q + 1) - 2 \\
&= \eta[CS_4(\mathcal{P}_4 : p, 1, q, 1)].
\end{aligned}$$

Similarly, we can calculate for all cases. We have summarized these results in a consolidated manner in Table 1.

When $q' \geq 2p' - ||\lfloor \frac{p'-2}{2} \rfloor||$, then note that

$$\begin{aligned}
q'' = q - q' - 1 &\leq q - 2p' + ||\lfloor \frac{p'-2}{2} \rfloor|| - 1 \\
&\leq q - 2p' + \frac{p'}{2} - 2 \\
&\leq q - 1.5(p - p'' - 1) - 2 \\
&\leq (q - 1.5p) + 1.5p'' - 0.5 \\
&< 1.5p'' - 1 \\
&\leq 2p'' - \lfloor \frac{p'' - 2}{2} \rfloor.
\end{aligned} \tag{15}$$

i	$\{i', i''\}$	$\eta[CS_4(\mathcal{P}_4 : p', 1, q', 1)]$ $\oplus$ $\eta[CS_4(\mathcal{P}_4 : p'', 1, q'', 1)]$	$\eta[CS_4(\mathcal{P}_4 : p, 1, q, 1)]$
5	$\{5,3\}$ $\{1,2\}$ $\{4,4\}$	$\frac{4}{5}(p+q)-6$ $\frac{4}{5}(p+q)-4$ $\frac{4}{5}(p+q)-4$	$\frac{4}{5}(p+q)-2$
4	$\{5,2\}$ $\{1,1\}$ $\{3,4\}$	$\frac{4}{5}(p+q+1)-6$ $\frac{4}{5}(p+q+1)-4$ $\frac{4}{5}(p+q+1)-6$	$\frac{4}{5}(p+q+1)-2$
3	$\{5,1\}$ $\{2,4\}$ $\{3,3\}$	$\frac{4}{5}(p+q+2)-6$ $\frac{4}{5}(p+q+2)-6$ $\frac{4}{5}(p+q+2)-8$	$\frac{4}{5}(p+q+2)-4$
2	$\{5,5\}$ $\{4,1\}$ $\{3,2\}$	$\frac{4}{5}(p+q+3)-8$ $\frac{4}{5}(p+q+3)-6$ $\frac{4}{5}(p+q+3)-8$	$\frac{4}{5}(p+q+3)-4$
1	$\{5,4\}$ $\{3,1\}$ $\{2,2\}$	$\frac{4}{5}(p+q+4)-8$ $\frac{4}{5}(p+q+4)-8$ $\frac{4}{5}(p+q+4)-8$	$\frac{4}{5}(p+q+4)-4$

Table 1: Comparison table.

Thus, if $p'' + q'' \equiv i'' \pmod 5$, we must have

$$\begin{aligned}
&\eta[CS_4(\mathcal{P}_4 : p', 1, q', 1)] \oplus \eta[CS_4(\mathcal{P}_4 : p'', 1, q'', 1)] \\
&\quad \le 2p' + \frac{4}{5}(p'' + q'' - i'') + 2\lfloor \frac{i''}{4} \rfloor \\
&\quad = 2p' + \frac{4}{5}((p - p' - 1) + (q - q' - 1) - i'') + 2\lfloor \frac{i''}{4} \rfloor \\
&\quad = 2p' + \frac{4}{5}(p+q) - \frac{4}{5}((p' + q') + 2 + i'') + 2\lfloor \frac{i''}{4} \rfloor \\
&\quad \le 2p' + \frac{4}{5}(p+q) - \frac{4}{5}((p' + 2p' - \lfloor \frac{p'-2}{2} \rfloor) + 2 + i'') + 2\lfloor \frac{i''}{4} \rfloor \\
&\quad \le 2p' + \frac{4}{5}(p+q) - \frac{4}{5}((p' + 2p' - \frac{p'}{2} + 1) + 2 + i'') + 2\lfloor \frac{i''}{4} \rfloor \\
&\quad \le \frac{4}{5}(p+q) - \frac{4}{5}(3 + i'') + 2\lfloor \frac{i''}{4} \rfloor \\
&\quad < \frac{4}{5}(p + q - i) + 2\lfloor \frac{i}{4} \rfloor
\end{aligned}$$

where $p + q \equiv i \pmod 5$. Note that, in the last step, strict inequality hold irrespective of the values of i and i''. □

Now it remains to show that for each $x \in \{1, 2, \cdots, \frac{4}{5}(p+q-i)+2\lfloor \frac{i}{4} \rfloor\}$ where $(p+q) \equiv i \pmod 5$ it is possible to find p' and q' such that

$$\eta[CS_4(\mathcal{P}_4 : p', 1, q', 1)] \oplus \eta[CS_4(\mathcal{P}_4 : p'', 1, q'', 1)] = x.$$

Lemma 5.7. *Suppose that $\eta[CS_4(\mathcal{P}_4 : n, 1, q, 1)]$ satisfy the formula given in the statement of Theorem 4.1 for all $n \leq p-1$ where $p < q < 2p - \lfloor \frac{p-2}{2} \rfloor$. Then there exists a child of the game $CS_4(\mathcal{P}_4 : p, 1, q, 1)$ in its game tree having nimber equal to l where $l \in \{1, 3, \cdots, \frac{4}{5}(p+q-i) + 2\lfloor \frac{i}{4} \rfloor - 1\}$ and $(p+q) \equiv i \pmod 5$.*

Proof. Let $\frac{4}{5}(p+q-i) + 2\lfloor \frac{i}{4} \rfloor = n_1$ and let $\eta[CS_4(\mathcal{P}_4 : p-1, 1, q-1, 1)] = n_2$. We will show that $n_2 \geq n_1 - 2$.

If $p - 1 < q - 1 < 2(p-1) - \lfloor \frac{p-3}{2} \rfloor$, then, assuming $(p+q-2) \equiv i_2 \pmod 5$, we have

$$\begin{aligned} n_2 - n_1 &= \frac{4}{5}(p-1+q-1-i_2) + 2\lfloor \frac{i_2}{4} \rfloor\} - \frac{4}{5}(p+q-i) + 2\lfloor \frac{i}{4} \rfloor \\ &= \frac{4}{5}(i - i_2 - 2) + 2\lfloor \frac{i_2}{4} \rfloor - 2\lfloor \frac{i}{4} \rfloor. \end{aligned}$$

Notice that, $i = 3, 4, 5$ implies $i_2 = i - 2$. Thus

$$n_2 - n_1 = 2\lfloor \frac{i_2}{4} \rfloor - 2\lfloor \frac{i}{4} \rfloor \geq -2.$$

If $i = 1, 2$, then $i_2 = i + 3$ which implies

$$n_2 - n_1 = -4 + 2\lfloor \frac{i_2}{4} \rfloor - 2\lfloor \frac{i}{4} \rfloor \geq -2.$$

If $2(p-1) - \lfloor \frac{p-3}{2} \rfloor \leq q - 1 < 2p - \lfloor \frac{p-2}{2} \rfloor$, then as the maximum difference between the lower and upper bound of $q-1$ is at most 1, and as $q-1$ is an integer, we can say that

$$q = 2(p-1) - \lfloor \frac{p-3}{2} \rfloor + 1.$$

Hence

$$\begin{aligned} n_2 - n_1 &= 2(p-1) - \frac{4}{5}(p+q-i) + 2\lfloor \frac{i}{4} \rfloor \\ &\geq 2p - 2 - \frac{4}{5}(p + 2p - 2 - \lfloor \frac{p-3}{2} \rfloor + 1 - i) + 2\lfloor \frac{i}{4} \rfloor \\ &\geq 2p - 2 - \frac{4}{5}(3p - 1 - (\frac{p-3}{2}) + 0.5 - i) + 2\lfloor \frac{i}{4} \rfloor \\ &= 2p - 2 - \frac{4}{5}(\frac{5p}{2} + 1 - i) + 2\lfloor \frac{i}{4} \rfloor \\ &= -2 - \frac{4}{5}(1 - i) + 2\lfloor \frac{i}{4} \rfloor \\ &\geq -2. \end{aligned}$$

Therefore, $n_2 \geq n_1 - 2$. That means, for each non-negative integer less than n_2 the game $CS_4(\mathcal{P}_4 : p-1, 1, q-1, 1)$ has a child having nimber equal to it. In particular, for each non-negative odd integer less than n_2, $CS_4(\mathcal{P}_4 : p-1, 1, q-1, 1)$ has a child having nimber equal to it.

Let $l \in \{1, 3, \cdots, n_1 - 3\}$ be an odd integer. We know that, $CS_4(\mathcal{P}_4 : p-1, 1, q-1, 1)$ has a child with nimber l. However, we also know that the child must be obtained by connecting x_{a+1} and y_{a+1} with a curve, and its nimber is given by

$$\eta[CS_4(\mathcal{P}_4 : a, 1, a, 1)] \oplus \eta[CS_4(\mathcal{P}_4 : p-a-2, 1, q-a-2, 1)] = l.$$

As $\eta[CS_4(\mathcal{P}_4 : a, 1, a, 1)] = 1$, we must have

$$\eta[CS_4(\mathcal{P}_4 : p-a-2, 1, q-a-2, 1)] = l - 1.$$

Thus the child of $CS_4(\mathcal{P}_4 : p, 1, q, 1)$ obtained by connecting x_{a+2} and y_{a+2} has nimber

$$\eta[CS_4(\mathcal{P}_4 : a+1, 1, a+1, 1)] \oplus \eta[CS_4(\mathcal{P}_4 : p-a-2, 1, q-a-2, 1)] = l.$$

Moreover, the child of $CS_4(\mathcal{P}_4 : p, 1, q, 1)$ obtained by connecting 1 and y_1 has nimber

$$\eta[CS_4(\mathcal{P}_4 : 0, 1, 0, 1)] \oplus \eta[CS_4(\mathcal{P}_4 : p-1, 1, q-1, 1)] = n_1 - 1.$$

Thus, for each $l \in \{1, 3, \cdots, n_1 - 1\}$ there exists a child of $CS_4(\mathcal{P}_4 : p, 1, q, 1)$ having nimber equal to l. $\square$

Last but not the least, we are going to show that the game $CS_4(\mathcal{P}_4 : p, 1, q, 1)$ has child having nimber equal to l for all even positive integers less than $\frac{4}{5}(p+q-i) + 2\lfloor \frac{i}{4} \rfloor\}$ where $(p+q) \equiv i \pmod 5$.

Lemma 5.8. *Suppose that $\eta[CS_4(\mathcal{P}_4 : n, 1, q, 1)]$ satisfy the formula given in the statement of Theorem 4.1 for all $n \leq p-1$ where $p < q < 2p - \lfloor \frac{p-2}{2} \rfloor$. Then there exists a child of the game $CS_4(\mathcal{P}_4 : p, 1, q, 1)$ in its game tree having nimber equal to l where $l \in \{2, 4, \cdots, \frac{4}{5}(p+q-i) + 2\lfloor \frac{i}{4} \rfloor - 2\}$ and $(p+q) \equiv i \pmod 5$.*

Proof. Let $\frac{4}{5}(p+q-i) + 2\lfloor \frac{i}{4} \rfloor = n_1$. We want to show that

$$\eta[CS_4(\mathcal{P}_4 : n, 1, q, 1)] = n_1.$$

To do so, we will consider some cases. The first case is if

$$(q+1) \geq 2(p-1) - \lfloor \frac{(p-1)-2}{2} \rfloor.$$

This implies

$$\begin{aligned} q - 1 &\geq 2(p-1) - \lfloor \frac{(p-1)-2}{2} \rfloor - 2 \\ &\geq 2(p-3) - \lfloor \frac{(p-3)-2}{2} \rfloor. \end{aligned}$$

Additionally, if p is even, then

$$\begin{aligned} q - 1 &\geq 2(p-1) - \lfloor \frac{(p-1)-2}{2} \rfloor - 2 \\ &\geq 2(p-2) - \lfloor \frac{(p-2)-2}{2} \rfloor. \end{aligned}$$

As we also know that

$$q < 2p - \lfloor \frac{p-2}{2} \rfloor.$$

Hence

$$\begin{aligned}
n_1 &= \frac{4}{5}(p+q-i) + 2\lfloor \frac{i}{4} \rfloor \\
&< \frac{4}{5}(p+2p-\lfloor \frac{p-2}{2} \rfloor - i) + 2\lfloor \frac{i}{4} \rfloor \\
\implies n_1 &\leq \frac{4}{5}(3p - \frac{p}{2} + 1.5 - i) + 2\lfloor \frac{i}{4} \rfloor - 1 \\
&\leq 2p + \frac{4}{5}(1.5-i) + 2\lfloor \frac{i}{4} \rfloor - 1 \\
\implies n_1 &< 2p \\
\implies n_1 &\leq 2(p-1).
\end{aligned}$$

In these cases, consider the child of $CS_4(\mathcal{P}_4 : p, 1, q, 1)$ which can be written as the sum of $CS_4(\mathcal{P}_4 : p', 1, q-1, 1)$ and $CS_4(\mathcal{P}_4 : p-p'-1, 1, 0, 1)$ where $q-1 \geq 2p' - \lfloor \frac{p'-2}{2} \rfloor$. Notice that

$$\eta[CS_4(\mathcal{P}_4 : p-p'-1, 1, 0, 1)] = 0$$

according to equation (5) and

$$\eta[CS_4(\mathcal{P}_4 : p', 1, q-1, 1)] = 2p'.$$

Observe that, this will cover all the even numbers up to $2(p-3)$, and up to $2(p-2)$ if p is even.

Therefore, we only need to think about the case when p is odd and $n_1 = 2(p-1)$. In this case, n_1 is divisible by 4. Consider the child obtained by connecting x_{p-1} with y_{q-2}. This will imply that $CS_4(\mathcal{P}_4 : p, 1, q, 1)$ has a child having nimber

$$\eta[CS_4(\mathcal{P}_4 : p-2, 1, q-3, 1)] \oplus \eta[CS_4(\mathcal{P}_4 : 1, 1, 2, 1)] = (n_1 - 4) \oplus 2 = n_1 - 2.$$

Thus we are done when $(q+1) \geq 2(p-1) - \lfloor \frac{(p-1)-2}{2} \rfloor$.

On the other hand, if $(q+1) < 2(p-1) - \lfloor \frac{(p-1)-2}{2} \rfloor$, then $p < q$ implies $p-1 < q+1$, by induction hypothesis we have $\eta[CS_4(\mathcal{P}_4 : p-1, 1, q+1, 1)] = n_1$.

Thus, in particular, for each $l \in \{2, 4, \cdots, n_1 - 2\}$, there exists a child of $CS_4(\mathcal{P}_4 : p-1, 1, q+1, 1)$ having nimber equal to l. Observe that, this child must be a sum of two games of the form $CS_4(\mathcal{P}_4 : p', 1, q', 1)$ and $CS_4(\mathcal{P}_4 : p-p'-2, 1, q-q', 1)$. Due to equation (15), we know that one of these child will satisfy the conditions of the second line of the formula given in Theorem 4.1.

Next consider the child of $CS_4(\mathcal{P}_4 : p, 1, q, 1)$ obtained by connecting $x_{p'+1}$ and $y_{q'+1}$ can be expressed as a sum of the games $CS_4(\mathcal{P}_4 : p', 1, q', 1)$ and $CS_4(\mathcal{P}_4 : p-p'-1, 1, q-q'-1, 1)$. Due to equation (15), we know that one of these child will satisfy the conditions of the second line of the formula given in Theorem 4.1.

In fact as $p \neq q$, it is possible to assume without loss of generality that both $CS_4(\mathcal{P}_4 : p-p'-2, 1, q-q', 1)$ and $CS_4(\mathcal{P}_4 : p-p'-1, 1, q-q'-1, 1)$ satisfy the conditions of the second line of the formula given in Theorem 4.1.

This implies

$$\eta[CS_4(\mathcal{P}_4 : p - p' - 2, 1, q - q', 1)] = \frac{4}{5}(p + q - p' - q' - 2 - i_1) + 2\lfloor\frac{i_1}{2}\rfloor$$
$$= \eta[CS_4(\mathcal{P}_4 : p - p' - 1, 1, q - q' - 1, 1)].$$

Thus, we can conclude that this game, expressed as a sum of $CS_4(\mathcal{P}_4 : p', 1, q', 1)$ and $CS_4(\mathcal{P}_4 : p - p' - 1, 1, q - q' - 1, 1)$, which is a child of $CS_4(\mathcal{P}_4 : p, 1, q, 1)$, has nimber equal to l. □

At last we are ready to prove Theorem 3.4.

Proof of Theorem 3.4. As $\eta[BS_2(\mathcal{P}_4 : p, q)] = 0$ for all $p, q \geq 3$, the second player must have a winning strategy. □

Using this we can calculate the nimber of the game $BS_2(\mathcal{P}_4 : p, q)$.

Corollary 5.9. *The nimber of the game $BS_2(\mathcal{P}_4 : p, q)$ is 0 for all $p, q \geq 3$.*

Proof. Observe that in the game tree of $BS_2(\mathcal{P}_4 : p, q)$, the root has exactly one child as the very first move is unique up to renaming of open tips. Moreover, that child will have a child which can be expressed as a sum of $CS_4(p, 1, 1, 1)$ and $CS_4(1, 1, q, 1)$. As $\eta[CS_4(t_1, 1, 1, 1)] \oplus \eta[CS_4(1, 1, t_3, 1)] = 0$ due to Theorem 4.1, the child of the root must have a non-zero nimber, and thus the root must have its nimber equal to 0. □

6 Conclusions

We studied two variants of the combinatorial game Sprouts, namely, the previously known Brussels sprouts - albeit a generalization introduced by us, and the Circular sprouts - a related game introduced as a tool to study a special case of Brussels sprouts. We ended up discovering that the game Circular sprouts is interesting on its own, and we propose to study it in general. Moreover, we also think that the game Circular sprouts is potentially a tool to attack the long standing Sprouts Conjecture.

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