

Homomorphisms of (n, m) -graphs with respect to generalized switch

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Abstract

An (n, m) -graph has n different types of arcs and m different types of edges. A homomorphism of an (n, m) -graph G to an (n, m) -graph H is a vertex mapping that preserves adjacency types and directions. Notice that, in an (n, m) -graph a vertex can possibly have $(2n + m)$ different types of neighbors. In this article, we study homomorphisms of (n, m) -graphs while an Abelian group acts on the set of different types of neighbors of a vertex.

Keywords: colored mixed graphs, switching, homomorphism, categorical product, chromatic number.

1 Introduction

A *graph homomorphism*, that is, an edge-preserving vertex mapping of a graph G to a graph H , also known as an H -coloring of G , was introduced as a generalization of coloring [5]. It allows us to unify certain important constraint satisfaction problems, especially related to scheduling and frequency assignments, which are otherwise expressed as various coloring and labeling problems on graphs. Thus the notion of graph homomorphism manages to capture a wide range of important applications in an uniform setup. When viewed as an operation on the set of all graphs, it induces rich algebraic structures: a quasiorder (and a partial order), a lattice, and a category.

The study of graph homomorphism can be characterized into three major branches:

1. The study of various application motivated optimization problems modeled using graph homomorphisms. These usually involve finding an H having certain prescribed properties such that every member of a graph family \mathcal{F} is H -colorable.

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2. The study of the algorithmic aspects of the H -coloring problem, including characterizing its dichotomy, and finding approximation or parametrized algorithms for the hard problems.
3. The study of the algebraic structures that gets induced by the notion of graph homomorphisms.

Unsurprisingly, these three areas of research have interdependencies and connections. The notion of graph homomorphisms, initially introduced for undirected and directed graphs, later got extended to 2-edge-colored graphs [11], k -edge-colored graphs [1] and (n, m) -colored mixed graphs [14]. These graphs, due to their various types of adjacencies, manages to capture complex relational structures and are useful for mathematical modeling. For instance, the query analysis problem in graph database, the databases that are now popularly used to handle highly interrelated data networks (such as, social networks like Facebook, Twitter, etc.), is modeled on homomorphisms of (n, m) -colored mixed graphs.

Moreover, researchers have started further extending the studies by exploring the effect of switch operation on homomorphisms. Notably, homomorphisms of signed graphs, which is essentially obtained by observing the effect of the switch operation on 2-edge-colored graphs, has gained immense popularity [13, 12, 16, 15] in recent times due to its strong connection with the graph minor theory. Also, graph homomorphism with respect to some other switch-like operations, such as, push operation on oriented graphs [8], cyclic switch on k -edge-colored graphs [3], and switching (n, m) -colored mixed graphs with respect to Abelian groups of special type (which does not allow switching an edge to an arc or vice versa) [4, 10] to list a few, has been recently studied.

Naturally, all the three main branches of research listed above in context of graph homomorphism is also explored for the above mentioned extensions and variants. However, in comparison, the global algebraic structure is a less explored branch for the extensions.

In this article, we are going to introduce a generalized switch operation on (n, m) -colored mixed graphs and study some of its algebraic aspects. The results proved here will be valid for all known graph homomorphism variants, to the best of our knowledge.

2 Homomorphisms of (n, m) -graphs and generalized switch

Throughout this article, we will follow the standard graph theoretic, algebraic and category theory notions from West [17], Artin [2], and Jacobson [6], respectively.

Nešetřil and Raspaud [14] introduced the concept of colored mixed graphs in 2000 as a generalization to the study of oriented and k -edge-colored graphs. An (n, m) -graph G is a graph with vertex set $V(G)$, arc set $A(G)$ and edge set $E(G)$, where each arc is colored with one of the n colors from $\{1, 2, \dots, n\}$ and each edge is colored with one of the m colors from $\{n+1, n+2, \dots, n+m\}$. In particular, if there is an arc of color i from u to v , we say that v is an i -neighbor of u , or equivalently, u is a $-i$ -neighbor of v . Furthermore, if there is an edge of color j between u and v , then we say that u (resp., v) is a j -neighbor of v (resp., u).

Let $\Gamma \subseteq S_{2n+m}$, where S_{2n+m} is the permutation group on $A_{n,m} = \{\pm 1, \dots, \pm n, n+1, \dots, n+m\}$. To Γ -switch a vertex v of an (n, m) -graph is to change its incident arcs and edges in such a way that its t -neighbors become $\sigma(t)$ -neighbors, for some $\sigma \in \Gamma$ and for all $t \in A_{n,m}$. For an element $\sigma \in S_{2n+m}$, a σ -switch of a vertex is defined similarly. An (n, m) -graph G' obtained by a sequence of Γ -switches performed on the vertices of G is a Γ -equivalent graph of G .

A Γ -homomorphism of G to H is a function $f : V(G) \rightarrow V(H)$ such that there exists a Γ -equivalent graph G' of G satisfying the following: if u is a t -neighbor of v in G' , then $f(u)$ is a t -neighbor of $f(v)$ in H . We denote this by $G \xrightarrow{\Gamma} H$. A Γ -isomorphism of G to H is a bijective Γ -homomorphism whose inverse is also a Γ -homomorphism. We denote this by $G \equiv_{\Gamma} H$.

Observe that, if $\Gamma = \langle e \rangle$ is the singleton group with the identity element e , then our Γ -homomorphism definition becomes the same as the colored homomorphism of (n, m) -graphs defined by Nešetřil and Raspaud [14].

We will restrict ourselves to Abelian subgroups Γ of S_{2n+m} unless otherwise stated. A similar study when Γ is non-Abelian is done in [7]. A *consistent* group $\Gamma \subset S_{2n+m}$ is such that each orbit induced by Γ acting on the set $A_{n,m}$ contains $-i$ if and only if it contains i for $i \in \{1, 2, \dots, n\}$.

3 Basic algebraic properties

Let $\Gamma \subseteq S_{2n+m}$ be a group and let G be an (n, m) -graph with set of vertices $\{v_1, v_2, \dots, v_k\}$. Let G^* be the (n, m) -graph having vertices of the type v_i^{σ} where $i \in \{1, 2, \dots, k\}$ and $\sigma \in \Gamma$. Also a vertex v_i^{σ} is a t -neighbor of $v_j^{\sigma'}$ in G^* if and only if v_i is a t -neighbor of v_j in G where $i, j \in \{1, 2, \dots, k\}$ and $\sigma, \sigma' \in \Gamma$. The Γ -switched graph $\rho_{\Gamma}(G)$ of G is the (n, m) -graph obtained from G^* by performing a σ -switch on v_i^{σ} for all $i \in \{1, 2, \dots, k\}$ and $\sigma \in \Gamma$.

This Γ -switch graph helps build a bridge between $\langle e \rangle$ -homomorphism and Γ -homomorphism of two (n, m) -graphs.

Proposition 3.1. *Let G, H be (n, m) -graphs. We have $G \xrightarrow{\Gamma} H$ if and only if $G \xrightarrow{\langle e \rangle} \rho_{\Gamma}(H)$.*

Proof. For the “only if” part of the proof, suppose $f : G \xrightarrow{\Gamma} H$. Notice that the inclusion $i : H \xrightarrow{\Gamma} \rho_{\Gamma}(H)$ is a Γ -homomorphism. Thus, the composition function $i \circ f$ is a Γ -homomorphism of G to $\rho_{\Gamma}(H)$.

For the “if” part of the proof, suppose $r : G \xrightarrow{\langle e \rangle} \rho_{\Gamma}(H)$ be a $\langle e \rangle$ -homomorphism. Let g_1, g_2, \dots, g_p be the vertices of G , and let h_1, h_2, \dots, h_q be the vertices of H . Also, let $r(g_i) = h_j^{\sigma_i}$ for some $\sigma_i \in \Gamma$. In that case, we define the function $\varphi : V(G) \rightarrow V(H)$ as follows:

$$\varphi(g_i) = h_j.$$

Now perform a σ_i^{-1} -switch on the vertex g_i of G to obtain its Γ -equivalent graph G' . Note that it is enough to show φ is a $\langle e \rangle$ -homomorphism of G' to H ,

Let g_i be a t -neighbor of g_j in G' . We have to show that $\varphi(g_i) = h_p$ is a t -neighbor of $\varphi(g_j) = h_q$ in H . Let σ_j, σ_i -switch is applied on g_i, g_j to obtain G' . This implies that g_i is a $\sigma_i(\sigma_j(t))$ -neighbor of g_j in G . Since r is a $\langle e \rangle$ -homomorphism, $h_p^{\sigma_i}$ is a $\sigma_i(\sigma_j(t))$ -neighbor of $h_q^{\sigma_j}$ in $\rho_\Gamma(H)$. Hence h_p is a $\sigma_i^{-1}(\sigma_j^{-1}(\sigma_i(\sigma_j(t))))$ -neighbor of h_q in H . As Γ is Abelian, we have

$$\sigma_i^{-1}(\sigma_j^{-1}(\sigma_i(\sigma_j(t)))) = t.$$

Hence h_p is a t -neighbor of h_q in H . \square

Theorem 3.2. *Let G, H be (n, m) -graphs. Then $G \equiv_\Gamma H$ if and only if $\rho_\Gamma(G) \equiv_{\langle e \rangle} \rho_\Gamma(H)$.*

Proof. For the “only if” part of the proof, suppose $f : G \xrightarrow{\Gamma} H$ is a Γ -isomorphism. That is, there exists some Γ -equivalent G' of G such that $f : G' \xrightarrow{\langle e \rangle} H$ is an $\langle e \rangle$ -isomorphism. Suppose that G has vertices g_1, g_2, \dots, g_p , and G' is obtained by performing τ_i -switch on g_i , for some $\tau_i \in \Gamma$.

For any $g_i^{\sigma_j} \in V(\rho_\Gamma(G))$ we define the function $\phi : V(\rho_\Gamma(G)) \rightarrow V(\rho_\Gamma(H))$ as follows:

$$\phi(g_i^{\sigma_j}) = \left(f(g_i)^{\tau_i^{-1}} \right)^{\sigma_j}.$$

Next we prove that ϕ is a $\langle e \rangle$ -isomorphism of $\rho_\Gamma(G)$ to $\rho_\Gamma(H)$. Note that $g_i^{\sigma_j}$ is a t -neighbor of $g_k^{\sigma_l}$ in $\rho_\Gamma(G)$ if and only if g_i is a $\sigma_j^{-1}(\sigma_l^{-1}(t))$ -neighbor of g_k in G . This is if and only if to the following: g_i is a $\tau_i(\tau_k(\sigma_j^{-1}(\sigma_l^{-1}(t))))$ -neighbor of g_k in G' . Since f is an $\langle e \rangle$ -isomorphism of G' to H , the previous statement is if and only if $f(g_i)$ is a $\tau_i(\tau_k(\sigma_j^{-1}(\sigma_l^{-1}(t))))$ -neighbor of $f(g_k)$ in H . This is true if and only if $\phi(g_i^{\sigma_j}) = \left(f(g_i)^{\tau_i^{-1}} \right)^{\sigma_j}$ is a $\gamma(t)$ -neighbor of $\phi(g_k^{\sigma_l}) = \left(f(g_k)^{\tau_k^{-1}} \right)^{\sigma_l}$ in $\rho_\Gamma(H)$ where

$$\gamma(t) = \sigma_l(\tau_k^{-1}(\sigma_j(\tau_i^{-1}(\tau_i(\tau_k(\sigma_j^{-1}(\sigma_l^{-1}(t)))))))).$$

However, as Γ is Abelian, $\gamma(t) = t$. Thus, ϕ a $\langle e \rangle$ -isomorphism of $\rho_\Gamma(G)$ to $\rho_\Gamma(H)$.

For the “if” part of the proof, suppose $\rho_\Gamma(G) \equiv_{\langle e \rangle} \rho_\Gamma(H)$ and we have to show $G \equiv_\Gamma H$.

Assume g_1, g_2, \dots, g_p be the vertices of G . A sequence of vertices in $\rho_\Gamma(G)$ of the form $(g_1^{\sigma_1}, g_2^{\sigma_2}, \dots, g_p^{\sigma_p})$ is a representative sequence of G in $\rho_\Gamma(G)$, where $\sigma_i \in \Gamma$ is any element for $i \in \{1, 2, \dots, p\}$ (repetition of elements among σ_i s is allowed here).

Given an $\langle e \rangle$ -isomorphism $\psi : \rho_\Gamma(G) \xrightarrow{\langle e \rangle} \rho_\Gamma(H)$ and a representative sequence S of G in $\rho_\Gamma(G)$, define the set

$$Y_{S,\psi} = \{v^\sigma \mid \psi(v^\sigma) = (\psi(v))^\sigma \text{ where } v \in S \text{ and } \sigma \in \Gamma\}.$$

Let $Y_{S^*,\varphi}$ be the set satisfying the property $|Y_{S^*,\varphi}| \geq |Y_{S,\psi}|$ where S varies over all representative sequences and ψ varies over all $\langle e \rangle$ -isomorphisms.

We will show that $Y_{S^*,\varphi} = V(\rho_\Gamma(G))$. We will prove by contradiction. Thus, let us assume the contrary, that is, let $Y_{S^*,\varphi} \neq V(\rho_\Gamma(G))$. This implies that there exists a v^σ , for some $v \in S^*$ and some $\sigma \in \Gamma$ such that $\varphi(v^\sigma) \neq (\varphi(v))^\sigma$. Next let us define the function

$$\hat{\varphi}(x) = \begin{cases} \varphi(x) & \text{if } x \neq g, v^\sigma, \\ \varphi(g) & \text{if } x = v^\sigma, \\ \varphi(v^\sigma) & \text{if } x = g, \end{cases}$$

where $g = \varphi^{-1}(\varphi(v)^\sigma) \in \rho_\Gamma(G)$.

Next we are going to show that $\hat{\varphi}$ is an $\langle e \rangle$ -isomorphism of $\rho_\Gamma(G)$ and $\rho_\Gamma(H)$. So, we need to show that x is a t -neighbor of y in $\rho_\Gamma(G)$ if and only if $\hat{\varphi}(x)$ is a t -neighbor of $\hat{\varphi}(y)$ in $\rho_\Gamma(H)$. Notice that, it is enough to check this for $x = g$ and $x = v^\sigma$ while y varies over all vertices of $\rho_\Gamma(G)$. We will separately handle the exceptional case when $x = v^\sigma$ and $y = g$ first.

- (i) When $x = v^\sigma$ and $y = g$: Note that $\varphi(v)$ and $\varphi(v)^\sigma$ are non-adjacent. Thus, $v = \varphi^{-1}(\varphi(v))$ and $g = \varphi^{-1}(\varphi(v)^\sigma)$ are non-adjacent. Hence $x = v^\sigma$ and $y = g$ are also non-adjacent. On the other hand, this implies that $\hat{\varphi}(x) = \varphi(g)$ and $\hat{\varphi}(y) = \varphi(v^\sigma)$ are non-adjacent.
- (ii) When $x = v^\sigma$ and $y \neq g$: Note that x is a t -neighbour of y in $\rho_\Gamma(G)$ if and only if $\varphi(x)$ is a t -neighbor of $\varphi(y)$ in $\rho_\Gamma(H)$, as φ is an $\langle e \rangle$ -isomorphism. Observe that $\hat{\varphi}(x) = \varphi(g) = \varphi(v)^\sigma$ as $g = \varphi^{-1}(\varphi(v)^\sigma)$, and $\hat{\varphi}(y) = \varphi(y)$. Since $x = v^\sigma$, v is a $\sigma^{-1}(t)$ -neighbor of y in $\rho_\Gamma(G)$, if and only if $\varphi(v)$ is a $\sigma^{-1}(t)$ -neighbor of $\varphi(y)$ in $\rho_\Gamma(H)$, if and only if, $\varphi(v)^\sigma = \hat{\varphi}(x)$ is a t -neighbour of $\varphi(y) = \hat{\varphi}(y)$. in $\rho_\Gamma(H)$.
- (iii) When $x = g$ and $y \neq v^\sigma$: Note that x is a t -neighbor of y in $\rho_\Gamma(G)$ if and only if, $\varphi(x) = \varphi(g) = \varphi(v)^\sigma$ is a t -neighbor of $\varphi(y)$ in $\rho_\Gamma(H)$, as $g = \varphi^{-1}(\varphi(v)^\sigma)$. The previous statement holds if and only if $\varphi(v)$ is $\sigma^{-1}(t)$ -neighbor of $\varphi(y)$ in $\rho_\Gamma(H)$ if and only if v is $\sigma^{-1}(t)$ -neighbor of y in $\rho_\Gamma(G)$ if and only if v^σ is t -neighbor of y in $\rho_\Gamma(G)$ if and only if $\varphi(v^\sigma) = \hat{\varphi}(x)$ is t -neighbor of $\varphi(y) = \hat{\varphi}(y)$ in $\rho_\Gamma(H)$.

However, now we have $|Y_{S^*, \varphi}| < |Y_{S^*, \hat{\varphi}}|$. This is a contradiction to the definition of $Y_{S^*, \varphi}$, and hence $Y_{S^*, \varphi} = V(\rho_\Gamma(G))$.

Let $v_1, v_2 \in S^*$. If $\varphi(v_1)^\sigma = \varphi(v_2)$ for any $\sigma \in \Gamma$, then $\varphi(v_1^\sigma) = \varphi(v_2)$. This implies $v_1^\sigma = v_2$ because φ is a bijection. However, this is not possible as v_1, v_2 are elements of the same representative sequence of G in $\rho_\Gamma(G)$. Hence, $\varphi(v_1)^\sigma \neq \varphi(v_2)$ for any $v_1, v_2 \in S^*$. That means, $\varphi(S^*) = R$ is a representative sequence of H in $\rho_\Gamma(H)$. Thus, note that $\langle e \rangle$ -isomorphism restricted to the induced subgraph $\rho_\Gamma(G)[S^*]$ is also an $\langle e \rangle$ -isomorphism to the induced subgraph $\rho_\Gamma(H)[R]$. That is, $\rho_\Gamma(G)[S^*] \equiv_{\langle e \rangle} \rho_\Gamma(H)[R]$. As $\langle e \rangle \subseteq \Gamma$, this also means $\rho_\Gamma(G)[S^*] \equiv_\Gamma \rho_\Gamma(H)[R]$.

On the other hand, as S^* and R are representative sequences of G and H , respectively, we have $\rho_\Gamma(G)[S^*] \equiv_\Gamma G$ and $\rho_\Gamma(H)[R] \equiv_\Gamma H$. Thus we are done by composing the Γ -isomorphisms. \square

The next result follows from the fundamental theorem of finite abelian groups.

Theorem 3.3. *Let Γ_1 be a consistent Abelian subgroup of S_{2n+m} . Let $\Gamma_2 \subseteq \Gamma_1$. If $p^2 \nmid |\Gamma_1|$ for any prime p , then $\rho_{\Gamma_1}(G) \equiv_{\langle e \rangle} \rho_{\Gamma_1/\Gamma_2}(\rho_{\Gamma_2}(G))$.*

Proof. Since Γ_1 is a finite Abelian group, Γ_1/Γ_2 and Γ_2 both are normal subgroups of Γ_1 , As, $p^2 \nmid |\Gamma_1|$, we have $\Gamma_1/\Gamma_2 \times \Gamma_2 \cong \Gamma_1$. Thus along with the fact that Γ_2 and Γ_1/Γ_2 are normal subgroups, we have $\Gamma_2 \cdot \Gamma_1/\Gamma_2 = \Gamma_1$. Thus we see that every element $\sigma \in \Gamma_1$, can be uniquely written as $\alpha.\beta$, where $\alpha \in \Gamma_1/\Gamma_2, \beta \in \Gamma_2$. Now let G be an (n, m) -graph. We prove, $f : \rho_{\Gamma_1}(G) \rightarrow \rho_{\Gamma_1/\Gamma_2}(\rho_{\Gamma_2}(G))$ is an isomorphism. Consider,

$$f : V(\rho_{\Gamma_1}(G)) \rightarrow V(\rho_{\Gamma_1/\Gamma_2}(\rho_{\Gamma_2}(G))),$$

$$f(u^\sigma) = (u^\alpha)^\beta.$$

where $\alpha.\beta = \sigma$.

Suppose, u^{σ_i} be a t -neighbor v^{σ_j} in $\rho_{\Gamma_1}(G)$ for some i, j if and only if $u^{\alpha_i.\beta_i}$ is a t -neighbor of $v^{\alpha_j.\beta_j}$, where $\sigma_i = \alpha_i.\beta_i$ and $\sigma_j = \alpha_j.\beta_j$. As Γ_2 and Γ_1/Γ_2 are Abelian, every $\sigma \in \Gamma_1$, can be uniquely represented as $\alpha.\beta$, where $\alpha \in \Gamma_1/\Gamma_2$ and $\beta \in \Gamma_2$. □

A Γ -core of an (n, m) -graph G is a subgraph H of G such that $G \xrightarrow{\Gamma} H$, whereas H does not admit a Γ -homomorphism to any of its proper subgraphs.

Theorem 3.4. *The core of an (n, m) -graph G is unique up to Γ -isomorphism.*

Proof. Let H_1 and H_2 be two Γ -cores of G . We have to show that H_1 and H_2 are Γ -isomorphic.

Note that, there exist Γ -homomorphisms $f_1 : G \xrightarrow{\Gamma} H_1$ and $f_2 : G \xrightarrow{\Gamma} H_2$ as H_1, H_2 are Γ -cores. Moreover, there exists inclusion Γ -homomorphisms $i_1 : H_1 \xrightarrow{\Gamma} G$ and $i_2 : H_2 \xrightarrow{\Gamma} G$.

Now consider the composition Γ -homomorphism $f_2 \circ i_1 : H_1 \xrightarrow{\Gamma} H_2$. Note that it must be a surjective vertex mapping. Not only that, for any non-adjacent pair u, v of vertices in H_1 , the vertices $(f_2 \circ i_1)(u)$ and $(f_2 \circ i_1)(v)$ are non-adjacent in H_2 . The reason is that, if the above two conditions are not satisfied, then the composition Γ -homomorphism $f_2 \circ i_1 \circ f_1 : G \xrightarrow{\Gamma} H_2$ can be considered as a Γ -homomorphism to a proper subgraph of H_2 . This will contradict the fact that H_2 is a Γ -core. Therefore, $f_2 \circ i_1$ is a bijective Γ -homomorphism whose inverse is also a Γ -homomorphism. In other words, $f_2 \circ i_1$ is a Γ -isomorphism. □

Due to the above theorem, it is possible to define the Γ -core of G and denote it by $core_\Gamma(G)$. Notice that, this is the analogue of the fundamental algebraic concept of core in the study of graph homomorphism.

4 Categorical products

Taking the set of (n, m) -graphs as objects and their Γ -homomorphisms as morphisms, one can consider the category of (n, m) -graphs with respect to Γ -homomorphism. In this section, we study whether products and co-products exist in this category or not. We would also like to remark that the existence of categorical product and co-product will not only contribute in establishing the category of (n, m) -graphs with respect to Γ -homomorphism as a richly structured category, but it will also show that the lattice of

(n, m) -graphs induced by Γ -homomorphisms is a distributive lattice with the categorical products and co-products playing the roles of join and meet, respectively. Moreover, categorical product was useful in proving the density theorem [5] for undirected and directed graphs. Thus, it is not wrong to hope that it may become useful to prove the analogue of the density theorem in our context.

Before proceeding further with the results, let us recall what categorical product and co-product mean in our context. Let G, H be two (n, m) -graphs and let $\Gamma \subseteq S_{2n+m}$ be an Abelian group.

The *categorical product* of G and H with respect to Γ -homomorphism is an (n, m) -graph P having two projection mappings of the form $f_g : P \xrightarrow{\Gamma} G$ and $f_h : P \xrightarrow{\Gamma} H$ satisfying the following universal property: if any (n, m) -graph P' admit Γ -homomorphisms $\phi_g : P' \xrightarrow{\Gamma} G$ and $\phi_h : P' \xrightarrow{\Gamma} H$, then there exists a unique Γ -homomorphism $\varphi : P' \xrightarrow{\Gamma} P$ such that $\phi_g = f_g \circ \varphi$ and $\phi_h = f_h \circ \varphi$.

The *categorical co-product* of G and H with respect to Γ -homomorphism is an (n, m) -graph C along with the two inclusion mappings of the form $i_g : G \xrightarrow{\Gamma} C$ and $i_h : H \xrightarrow{\Gamma} C$ satisfying the following universal property: if for any (n, m) -graph C' there are Γ -homomorphisms $\phi_g : G \xrightarrow{\Gamma} C'$ and $\phi_h : H \xrightarrow{\Gamma} C'$, then there exists a unique Γ -homomorphism $\varphi : C \xrightarrow{\Gamma} C'$ such that $\phi_g = \varphi \circ i_g$ and $\phi_h = \varphi \circ i_h$.

Let G, H be two (n, m) -graphs and let $\Gamma \subseteq S_{2n+m}$ be an Abelian group. Then $G \times_{\langle e \rangle} H$ denotes the (n, m) -graph on set of vertices $V(G) \times V(H)$ where (u, v) is a t -neighbor of (u', v') in $G \times_{\langle e \rangle} H$ if and only if u is a t -neighbor of u' in G and v is a t -neighbor of v' in H . Moreover, the (n, m) -graph $G \times_{\Gamma} H$ is the subgraph of $\rho_{\Gamma}(G) \times_{\langle e \rangle} \rho_{\Gamma}(H)$ induced by the set of vertices

$$X = \{(u^{\sigma}, v^{\sigma}) : (u, v) \in V(G) \times V(H) \text{ and } \sigma \in \Gamma\}.$$

Theorem 4.1. *The categorical product of (n, m) -graphs G and H with respect to Γ -homomorphism exists and is Γ -isomorphic to $G \times_{\Gamma} H$.*

Proof. Let $(G \times_{\Gamma} H)'$ be Γ -switched graph of $G \times_{\Gamma} H$, where we apply σ^{-1} on $(u^{\sigma}, v^{\sigma}) \in V(G \times_{\Gamma} H)$. Thus, we define $f_g(u^{\sigma}, v^{\sigma}) = u$ and $f_h(u^{\sigma}, v^{\sigma}) = v$ as the two projections. Observe that f_g and f_h are $\langle e \rangle$ -homomorphisms of $(G \times_{\Gamma} H)'$ to G and H , respectively. If there exists an (n, m) -graph P' such that, $\phi_g : P' \xrightarrow{\Gamma} G$ and $\phi_h : P' \xrightarrow{\Gamma} H$, then define $\phi : P' \xrightarrow{\Gamma} G \times_{\Gamma} H$ such that $\phi(p) = (\phi_g(p), \phi_h(p))$. From the definition of ϕ , we have $\phi_g = f_g \circ \phi$ and $\phi_h = f_h \circ \phi$. Note that this is the unique way we can define ϕ which satisfies the universal property from the definition of products. Thus, $G \times_{\Gamma} H$ is indeed the categorical product of G and H with respect to Γ -homomorphism once we prove its uniqueness up to Γ -isomorphism.

Suppose P_1 with homomorphisms f_g, f_h and P_2 with homomorphisms ϕ_g, ϕ_h be two (n, m) -graphs that satisfy the universal properties of categorical product of G and H , then there exists $\varphi : P_1 \xrightarrow{\Gamma} P_2$ and $\varphi' : P_2 \xrightarrow{\Gamma} P_1$ with $\phi_g \circ \varphi = f_g$, $\phi_h \circ \varphi = f_h$ and $f_g \circ \varphi' = \phi_g$, $f_h \circ \varphi' = \phi_h$. Now consider the composition, $\varphi' \circ \varphi : P_1 \xrightarrow{\Gamma} P_1$. As, $f_g \circ (\varphi' \circ \varphi) = f_g$, $f_h \circ (\varphi' \circ \varphi) = f_h$, we should have $\varphi' \circ \varphi$ to be the identity mapping on P_1 . Similarly $\varphi \circ \varphi'$ must be the identity mapping on P_2 . Thus implying, $\varphi' = \varphi^{-1}$ is an Γ -isomorphism of P_2 and P_1 . \square

Corollary 4.2. *From the Theorem 3.2 and Theorem 4.1, we have, $\rho_\Gamma(G \times_\Gamma H) \equiv_{\langle e \rangle} \rho_\Gamma(G) \times_{\langle e \rangle} \rho_\Gamma(H)$.*

Let $G + H$ denotes the disjoint union of the (n, m) -graphs G and H .

Theorem 4.3. *The categorical co-product of (n, m) -graphs G and H with respect to Γ -homomorphism exists and is Γ -isomorphic to $G + H$.*

Proof. Consider the inclusion mapping $i_g : G \xrightarrow{\langle e \rangle} G + H$, and $i_h : H \xrightarrow{\langle e \rangle} G + H$. Suppose there exists an (n, m) -graph C and there are Γ -homomorphisms $\phi_g : G \xrightarrow{\Gamma} C$ and $\phi_h : H \xrightarrow{\Gamma} C$, then there exists a Γ -homomorphism $\varphi : G + H \xrightarrow{\Gamma} C$ such that

$$\varphi(x) = \begin{cases} \phi_g(x) & \text{if } x \in V(G), \\ \phi_h(x) & \text{if } x \in V(H). \end{cases}$$

Observe that, $\phi_g = \varphi \circ i_g$ and $\phi_h = \varphi \circ i_h$. Note that such a φ is unique.

Suppose we have P with Γ -homomorphisms f_g, f_h and P' with Γ -homomorphisms ϕ_g, ϕ_h satisfying the universal property of categorical co-product of G and H , then there exists Γ -homomorphisms, $\varphi : P \xrightarrow{\Gamma} P'$ and $\varphi' : P' \xrightarrow{\Gamma} P$ with $\phi_g = \varphi \circ f_g$, $\phi_h = \varphi \circ f_h$ and $f_g = \varphi' \circ \phi_g$, $f_h = \varphi' \circ \phi_h$. Now consider the composition, $\varphi' \circ \varphi : P \xrightarrow{\Gamma} P$. As, $(\varphi' \circ \varphi) \circ f_g = f_g$, $(\varphi' \circ \varphi) \circ f_h = f_h$, we should have $\varphi' \circ \varphi$ to be the identity mapping on P . Similarly $\varphi \circ \varphi'$ must be the identity mapping on P' . Thus implying, $\varphi' = \varphi^{-1}$ is an Γ -isomorphism of P' and P . Therefore, we have, the categorical co-product of (n, m) -graphs G and H with respect to Γ -homomorphism is $G + H$. \square

Thus both categorical product and co-product exists with respect to Γ -homomorphism. Furthermore, the usual algebraic identities hold with respect to these operations too.

Theorem 4.4. *For any (n, m) -graphs G, H, K we have the following.*

- (i) $G \times_\Gamma H \equiv_\Gamma H \times_\Gamma G$,
- (ii) $G \times_\Gamma (H \times_\Gamma K) \equiv_\Gamma (G \times_\Gamma H) \times_\Gamma K$,
- (iii) $G \times_\Gamma (H + K) \equiv_\Gamma (G \times_\Gamma H) + (G \times_\Gamma K)$.

Proof. (i) In Theorem 4.1, we showed the existence and uniqueness (up to Γ -isomorphism) of the categorical product of G and H with respect to Γ -homomorphism. However, if one follows the definition of categorical product in this context, there is no distinction due to the order in which we consider G and H . Therefore, the categorical product of G and H will be the same as the categorical product of H and G . As these two categorical products are Γ -isomorphic to $G \times_\Gamma H$ and $H \times_\Gamma G$ due to Theorem 4.1, respectively, we are done.

(ii) Observe that when $\Gamma = \langle e \rangle$. the function

$$\phi(g, (h, k)) = ((g, h), k)$$

is a $\langle e \rangle$ -isomorphism of $G \times_\Gamma H$ to $H \times_\Gamma G$ where $g \in G$, $h \in H$, and $k \in K$.

Next we will prove it for general Γ . Notice that by Theorem 4.1, we have

$$\rho_\Gamma(G \times_\Gamma (H \times_\Gamma K)) = \rho_\Gamma(G) \times_{\langle e \rangle} \rho_\Gamma(H \times_\Gamma K) = \rho_\Gamma(G) \times_{\langle e \rangle} (\rho_\Gamma(H) \times_{\langle e \rangle} \rho_\Gamma(K)) \quad (1)$$

and

$$\rho_\Gamma((G \times_\Gamma H) \times_\Gamma K) = \rho_\Gamma(G \times_\Gamma H) \times_{\langle e \rangle} \rho_\Gamma(K) = (\rho_\Gamma(G) \times_{\langle e \rangle} \rho_\Gamma(H)) \times_{\langle e \rangle} \rho_\Gamma(K). \quad (2)$$

Notice that, as we have already proved that our inequality holds for $\Gamma = \langle e \rangle$, we know that

$$\rho_\Gamma(G) \times_{\langle e \rangle} (\rho_\Gamma(H) \times_{\langle e \rangle} \rho_\Gamma(K)) \equiv_{\langle e \rangle} (\rho_\Gamma(G) \times_{\langle e \rangle} \rho_\Gamma(H)) \times_{\langle e \rangle} \rho_\Gamma(K).$$

Therefore by equations (3) and (4) we have

$$\rho_\Gamma(G \times_\Gamma (H \times_\Gamma K)) \equiv_{\langle e \rangle} \rho_\Gamma((G \times_\Gamma H) \times_\Gamma K).$$

By Theorem 3.2 we have

$$G \times_\Gamma (H \times_\Gamma K) \equiv_\Gamma (G \times_\Gamma H) \times_\Gamma K.$$

This concludes the proof.

(iii) When $\Gamma = \langle e \rangle$ consider the the function

$$\phi(g, x) = (g, x)$$

where $g \in G$ and if $x \in (H + K)$. However, here if $x \in H$, then the image $(g, x) \in G \times_\Gamma H$ and if $x \in K$, then the image $(g, x) \in G \times_\Gamma K$. Observe that ϕ is a $\langle e \rangle$ -isomorphism of $G \times_\Gamma (H + K)$ to $(G \times_\Gamma H) + (G \times_\Gamma K)$.

Next we will prove it for general Γ . Notice that by Theorem 4.1, we have

$$\rho_\Gamma(G \times_\Gamma (H + K)) = \rho_\Gamma(G) \times_{\langle e \rangle} \rho_\Gamma(H + K) = \rho_\Gamma(G) \times_{\langle e \rangle} (\rho_\Gamma(H) + \rho_\Gamma(K)) \quad (3)$$

and

$$\begin{aligned} \rho_\Gamma((G \times_\Gamma H) + (G \times_\Gamma K)) &= \rho_\Gamma(G \times_\Gamma H) + \rho_\Gamma(G \times_\Gamma K) \\ &= (\rho_\Gamma(G) \times_{\langle e \rangle} \rho_\Gamma(H)) + (\rho_\Gamma(G) \times_{\langle e \rangle} \rho_\Gamma(K)). \end{aligned} \quad (4)$$

Notice that, as we have already proved that our inequality holds for $\Gamma = \langle e \rangle$, we know that

$$\rho_\Gamma(G) \times_{\langle e \rangle} (\rho_\Gamma(H) + \rho_\Gamma(K)) \equiv_{\langle e \rangle} (\rho_\Gamma(G) \times_{\langle e \rangle} \rho_\Gamma(H)) + (\rho_\Gamma(G) \times_{\langle e \rangle} \rho_\Gamma(K)).$$

Therefore by equations (3) and (4) we have

$$\rho_\Gamma(G \times_\Gamma (H + K)) \equiv_{\langle e \rangle} \rho_\Gamma((G \times_\Gamma H) + (G \times_\Gamma K)).$$

By Theorem 3.2 we have

$$G \times_\Gamma (H + K) \equiv_\Gamma (G \times_\Gamma H) + (G \times_\Gamma K).$$

This concludes the proof. □

5 Chromatic number

We know that the ordinary chromatic number of a simple graph G can be expressed as the minimum $|V(H)|$ such that G admits a homomorphism to H . The analogue of this definition is a popular way for defining chromatic number of other types of graphs, namely, oriented graphs, k -edge-colored graphs, (n, m) -graphs, signed graphs, push graphs, etc. Here also, we can follow the same.

The Γ -chromatic number of an (n, m) -graph is given by

$$\chi_{\Gamma:n,m}(G) = \min\{|V(H)| : G \xrightarrow{\Gamma} H\}.$$

Moreover, for a family \mathcal{F} of (n, m) -graphs, the Γ -chromatic number is given by

$$\chi_{\Gamma:n,m}(\mathcal{F}) = \max\{\chi_{\Gamma:n,m}(G) : G \in \mathcal{F}\}.$$

Let $\Gamma \subseteq S_{2n+m}$ be an Abelian group acting on the set $A_{n,m}$. For $x \in A_{n,m}$, we call the set, $\text{Orb}_x = \{\sigma(x) : \sigma \in \Gamma\}$ as *orbit of x* . Notice that, these orbits form a partition on the set $A_{n,m}$ as the relation, $x \sim y$ whenever $x = \sigma(y)$ for some $\sigma \in \Gamma$, is an equivalence relation. We present an important observation.

Proposition 5.1. *Let $\Gamma \subseteq S_{2n+m}$ be a consistent group, G be a (n, m) -graph and G' be a Γ -equivalent graph of G . If a vertex u is a t -neighbor of v in G , then u must be a $\sigma(t)$ neighbor of v in G' for some $\sigma \in \Gamma$.*

Proof. As Γ is a consistent group, there exists some $\sigma_i \in \Gamma$ such that $\sigma_i(\alpha) = -\alpha$ for $\alpha \in \{1, 2, 3, \dots, n\}$. Suppose u is a t -neighbor of v in G , and suppose σ_i is applied on u and σ_j is applied on v to obtain G' .

Case 1 : If $\sigma_i(t), \sigma_j(\sigma_i(t)) \in \{n+1, n+2, \dots, n+m\}$, then v is a $\sigma_j(\sigma_i(t)) = \sigma_k(t)$ -neighbor of u , where $\sigma_k = \sigma_j \cdot \sigma_i$. Also u is a $\sigma_j(\sigma_i(t)) = \sigma_k(t)$ -neighbor of v in G' . Suppose, if $\sigma_j(\sigma_i(t)) \in \{1, 2, \dots, n\}$, in this case v is $\sigma_j(\sigma_i(t))$ -neighbor of u , whereas u is $-\sigma_j(\sigma_i(t))$ -neighbor of v . That is, u is $-\sigma_k(t)$ -neighbor of v . Since Γ is consistent, there exists $\sigma_l \in \Gamma$ such that u is $\sigma_l(t)$ -neighbor of v .

Case 2 : If $\sigma_i(t) \in \{1, 2, \dots, n\}$, then u is a $\sigma_i(t)$ -neighbor of v , and, v is a $-\sigma_i(t)$ -neighbor of u . As Γ is consistent, there exists $\sigma_l \in \Gamma$ such that $\sigma_l(t) = -\sigma_i(t)$. Thus, v is a $\sigma_l(t)$ -neighbor of u . Now σ_j is applied on v . If $\sigma_j(\sigma_l(t)) \in \{n+1, n+2, \dots, n+m\}$, then, v is $\sigma_j(\sigma_l(t))$ -neighbor of u , and u also is $\sigma_j(\sigma_l(t))$ -neighbor of v . Let, $\sigma_r = \sigma_j \cdot \sigma_l$. Then, u is a $\sigma_r(t)$ -neighbor of v . If $\sigma_j(\sigma_l(t)) \in \{1, 2, \dots, n\}$, then v is $\sigma_j(\sigma_l(t)) = \sigma_k(t)$ -neighbor of u for some k , whereas u is $-\sigma_j(\sigma_l(t))$ -neighbor of v . Then we have, u is $-\sigma_j(\sigma_l(t)) = -\sigma_k(t)$ -neighbor of v , as Γ is consistent, $-\sigma_k(t) = \sigma_r(t)$ for some r . □

Next we focus on studying the Γ -chromatic number of (n, m) -forests.

Theorem 5.2. *Let \mathcal{F} be the family of (n, m) -forests and let t be the number of orbits of $A_{n,m}$ with respect to the action of Γ . Then,*

$$\chi_{\Gamma:n,m}(\mathcal{F}) \leq \begin{cases} t+2 & \text{if } t \text{ is even,} \\ t+1 & \text{if } t \text{ is odd.} \end{cases}$$

Moreover, equality holds if Γ is consistent.

Proof. We will start by proving the upper bound. First assume that t is odd. In this case consider the complete graph K_{t+1} . We will construct a complete (n, m) -graph having K_{t+1} as its underlying graph. As $(t + 1)$ is even, we know that K_{t+1} can be decomposed into $\frac{t-1}{2}$ edge-disjoint Hamiltonian cycles and a perfect matching.

Let $\{\alpha_1, \alpha_2, \dots, \alpha_t\}$ be the representatives of the t orbits. If $v_0 v_1 \dots v_t v_0$ is the i^{th} Hamiltonian cycle from the decomposition, then assign adjacencies to the edges of it in such a way that v_j is a α_{2j-1} -neighbor of v_{j-1} and v_{j+1} is a α_{2j} -neighbor of v_j , for $j \in \{0, 1, \dots, \frac{t-1}{2}\}$ where the $+$ operation of the indices is considered modulo $(t+1)$. Moreover, let $u_0 w_0, u_1 w_1, \dots, u_{\frac{t-1}{2}} w_{\frac{t-1}{2}}$ be the edges of the perfect matching from the decomposition mentioned above. Assign adjacencies to these edges in such a way that u_j is a α_t -neighbor of w_j for all $j \in \{0, 1, \dots, \frac{t-1}{2}\}$. With a little abuse of notation, we denote the so obtained (n, m) -graph by K_{t+1} itself.

Notice that, each vertex of K_{t+1} have a α_j -neighbor for all $i \in \{1, 2, \dots, t\}$. We claim that every (n, m) -forest admit a Γ -homomorphism to K_{t+1} . If not, then there exists a minimal (with respect to number of vertices) counter-example F that does not admit Γ -homomorphism to K_{t+1} . Let u be a leaf, having v as its neighbor, of F , then $F \setminus \{u\}$ is no longer a minimal counter-example, thus it admits a Γ -homomorphism f to K_{t+1} . That means, there exists a Γ -equivalence (n, m) -graph $F' \setminus \{u\}$ such that f is a $\langle e \rangle$ -homomorphism of it to K_{t+1} . Also assume that, F' is such a Γ -equivalent (n, m) -graph of F , that v is a α_i -neighbor of u for some $i \in \{1, 2, \dots, t\}$. This is possible as one can switch the vertex v to make the adjacency of u with v match the corresponding orbit's representative. Now, we extend f to a $\langle e \rangle$ -homomorphism of F' to K_{t+1} by mapping v to the α_i -neighbor $f(u)$ in K_{t+1} . That means, there exists a Γ -homomorphism of F' to K_{t+1} . This contradicts the minimality of F . Hence every (n, m) -forest admits a Γ -homomorphism to $K_{n,m}$.

Secondly, assume that t is even. Note that, if there were $t + 1$ orbits instead, then by what we have proved above, it was possible to show that all (n, m) -forests will admit a Γ -homomorphism to an (n, m) -graph having K_{t+2} as underlying graph. Therefore, assuming a dummy orbit we are done with this case too.

Next we will prove the tightness of the upper bound when Γ is consistent. Let $\{\alpha_1, \alpha_2, \dots, \alpha_t\}$ be the representatives of the t orbits. For odd values of t , consider the star (n, m) -graph S on $t + 1$ vertices: the central vertex v having t neighbors v_1, v_2, \dots, v_t . Let v_i be a α_i -neighbor of v for all $i \in \{1, 2, \dots, t\}$. As, no matter how we switch the vertices of S , the vertex v will have t distinctly adjacent neighbors. Therefore we have $\chi_{\Gamma, n, m}(S) \geq t + 1$, and thus $\chi_{\Gamma, n, m}(\mathcal{F}) = t + 1$ when t is odd and Γ is consistent.

For even values of t , consider a rooted tree T of height two in which every vertex, other than the leaves, has exactly one α_i -neighbor for $i \in \{1, 2, \dots, t\}$. Suppose T admits a Γ -homomorphism f to an (n, m) -graph H . Let r be the root of T . If H has $(t + 1)$ vertices, then the images of the vertices from $N[r]$ under f will be a spanning subgraph in H . Furthermore, notice that each vertex of $N[r]$ has at least one β_i -neighbor, where β_i belongs to the i^{th} orbit. Thus their images should also have the same property, that is each of them must have at least one β_i -neighbor, where β_i belongs to the i^{th} orbit. However, as H has only $(t + 1)$ vertices, each of its vertices are forced to have exactly one β_i -neighbor, where β_i belongs to the i^{th} orbit. Now if we restrict ourselves to only the

neighbors whose type is from a particular orbit, that must give us a perfect matching, which is impossible as $(t + 1)$ is odd. Therefore, H must have at least $(t + 2)$ vertices which implies the lower bound. \square

6 Concluding remarks

In this article, we introduced a generalized switch operation on (n, m) -graphs and studied its basic algebraic properties. This topic will generate plenty of natural open questions, especially, in an effort of extending the known results in the domain of graph homomorphisms. As a remark, using the notion of generalized switch (implicitly), it was possible to improve the upper bound of the $\langle e \rangle$ -chromatic number of (n, m) -partial 2-trees where $2n + m = 3$ [9].

References

- [1] N. Alon and T. H. Marshall. Homomorphisms of edge-colored graphs and coxeter groups. *Journal of Algebraic Combinatorics*, 8(1):5–13, 1998.
- [2] M. Artin. *Algebra*. Birkhäuser, 1998.
- [3] R. C. Brewster and T. Graves. Edge-switching homomorphisms of edge-coloured graphs. *Discrete mathematics*, 309(18):5540–5546, 2009.
- [4] R. C. Brewster, A. Kidner, and G. MacGillivray. k -colouring (m, n) -mixed graphs with switching. *arXiv preprint arXiv:2203.08070*, 2022.
- [5] P. Hell and J. Nešetřil. *Graphs and homomorphisms*, volume 28. OUP Oxford, 2004.
- [6] N. Jacobson. *Basic algebra I*. Courier Corporation, 2012.
- [7] A. Kidner. *Gamma-Switchable 2-Colourings of (m, n) -Mixed Graphs*. PhD thesis, 2021.
- [8] W. F. Klostermeyer and G. MacGillivray. Homomorphisms and oriented colorings of equivalence classes of oriented graphs. *Discrete Mathematics*, 274(1-3):161–172, 2004.
- [9] A. Lahiri, S. Nandi, S. Taruni, and S. Sen. On chromatic number of (n, m) -graphs. In *Extended Abstracts EuroComb 2021*, pages 745–751. Springer, 2021.
- [10] E. Leclerc, G. MacGillivray, and J. M. Warren. Switching (m, n) -mixed graphs with respect to abelian groups. *arXiv preprint arXiv:2110.01576*, 2021.
- [11] A. Montejano, P. Ochem, A. Pinlou, A. Raspaud, and É. Sopena. Homomorphisms of 2-edge-colored graphs. *Discrete Applied Mathematics*, 158(12):1365–1379, 2010.
- [12] R. Naserasr, E. Rollová, and É. Sopena. Homomorphisms of signed graphs. *Journal of Graph Theory*, 79(3):178–212, 2015.
- [13] R. Naserasr, E. Sopena, and T. Zaslavsky. Homomorphisms of signed graphs: An update. *European Journal of Combinatorics*, 91:103222, 2021.

- [14] J. Nešetřil and A. Raspaud. Colored homomorphisms of colored mixed graphs. *Journal of Combinatorial Theory, Series B*, 80(1):147–155, 2000.
- [15] P. Ochem, A. Pinlou, and S. Sen. Homomorphisms of signed planar graphs. *arXiv preprint arXiv:1401.3308*, 2014.
- [16] E. Sopena, E. Rollová, and R. Naserasr. Homomorphisms of planar signed graphs to signed projective cubes. *Discrete Mathematics & Theoretical Computer Science*, 15, 2013.
- [17] D. B. West. *Introduction to graph theory*, volume 2. Prentice hall Upper Saddle River, 2001.