A linear algorithm for radio k-coloring powers of paths having small diameter

Dipayan Chakraborty[1], Soumen Nandi[2], Sagnik Sen[3], and D K Supraja[2,3]

[1] Université Clermont-Auvergne, CNRS, Mines de Saint-Étienne, Clermont-Auvergne-INP, LIMOS, 63000 Clermont-Ferrand, France
dipayan.chakraborty@uca.fr
[2] Netaji Subhas Open University, India
soumen2004@gmail.com
[3] Indian Institute of Technology Dharwad, India
sen007isi@gmail.com, dksupraja95@gmail.com

Abstract. The radio k-chromatic number $rc_k(G)$ of a graph G is the minimum integer λ such that there exists a function $\phi : V(G) \to \{0, 1, \cdots, \lambda\}$ satisfying $|\phi(u) - \phi(v)| \geq k + 1 - d(u, v)$, where $d(u, v)$ denotes the distance between u and v. To date, several upper and lower bounds of $rc_k(\cdot)$ is established for different graph families. One of the most notable works in this domain is due to Liu and Zhu [SIAM Journal on Discrete Mathematics 2005] whose main results were computing the exact values of $rc_k(\cdot)$ for paths and cycles for the specific case when k is equal to the diameter.

In this article, we find the exact values of $rc_k(G)$ for powers of paths where the diameter of the graph is strictly less than k. Our proof readily provides a linear time algorithm for providing such a labeling. Furthermore, our proof technique is a potential tool for solving the same problem for other classes of graphs having "small" diameter.

Keywords: radio coloring · radio k-chromatic number · Channel Assignment Problem · power of paths.

1 Introduction and main results

The theory of radio coloring and its variations are popular and well-known mathematical models of the Channel Assignment Problem (CAP) in wireless networks [1, 2]. The connection between the real-life problem and the theoretical model has been explored in different bodies of works. In this article, we focus on the theoretical aspects of a particular variant, namely, the radio k-coloring. All the graphs considered in this article are undirected simple graphs and we refer to the book "Introduction to graph theory" by West [14] for all standard notations and terminologies used.

A *λ-radio k-coloring* of a graph G is a function $\phi : V(G) \to \{0, 1, \cdots, \lambda\}$ satisfying $|\phi(u) - \phi(v)| \geq k + 1 - d(u, v)$. For every $u \in V(G)$, the value $\phi(u)$ is generally referred to as the *color* of u under ϕ. Usually, we pick λ in such a way that it has a preimage under ϕ, and then, we call λ to be the span of ϕ, denoting it by $span(\phi)$. The *radio k-chromatic number*[4] $rc_k(G)$ is the minimum $span(\phi)$, where ϕ varies over all radio k-colorings of G.

In particular, the radio 2-chromatic number is the most well-studied restriction of the parameter (apart from the radio 1-chromatic number, which is equivalent to studying the chromatic number of graphs). There is a famous conjecture by Griggs and Yeh [6] that claims $rc_2(G) \leq \Delta^2$ where Δ is the maximum degree of G. The conjecture have been resolved for all $\Delta \geq 10^{69}$ by Havet, Reed and Sereni [7].

As one may expect, finding the exact values of $rc_k(G)$ for a general graph is an NP-complete problem [6]. Therefore, finding the exact value of $rc_k(G)$ for a given graph (usually belonging to a particular graph family) offers a huge number of interesting problems. Unfortunately, due to a lack of general techniques for solving these problems, not many exact values are known till date. One of the best contributions in this front remains the work of Liu and Zhu [12] who computed the exact value of $rc_k(G)$ where G is a path or a cycle and $k = diam(G)$.

As our work focuses on finding radio k-chromatic number of powers of paths, let us briefly recall the relevant related works. For a detailed overview of the topic, we encourage the reader to consult Chapter 7.5 of the dynamic survey on this topic maintained in the Electronic Journal of Combinatorics by Gallian [5] and the survey by Panigrahi [13]. For small paths P_n, that is, with $diam(P_n) < k$, Kchikech et al. [8] had established an exact formula for $rc_k(P_n)$; whereas, recall that, for paths of diameter equal to $k \geq 2$, Liu and Zhu [12] gave an exact formula for the radio number $rc_k(P_k)$. Moreover, a number of studies on the parameter $rc_k(P_n)$ depending on how k is related to $diam(P_n)$, or n alternatively, have been done by various authors [8–10, 3]. So far as works on powers of paths is concerned, the only notable work we know is an exact formula for the radio number $rn(P_n^2)$ of the square of a path P_n by Liu and Xie [11]. Hence the natural question to ask is whether the results for the paths can be extended to paths of a general power m, where $1 \leq m \leq n$.

Progressing along the same line, in this article we concentrate on powers of paths having "small diameters", that is, $diam(P_n^m) < k$ and compute the exact value of $rc_k(P_n^m)$, where P_n^m denotes the m-th power graph of a path P_n on $(n+1)$ vertices. In other words, the graph P_n^m is obtained by adding edges between the vertices of P_n that are at most m distance apart. Notice that, the so-obtained graph is, in particular, an interval graph. Let us now state our main theorem.

Theorem 1. *For all $k > diam(P_n^m)$, we have*

[4] In the case that $diam(G) = k, k+1$ and $k+2$, the radio k-chromatic number is alternatively known as the *radio number* denoted by $rn(G)$, the *radio antipodal number* denoted by $ac(G)$ and the *nearly antipodal number* denoted by $ac'(G)$, respectively.

$$rc_k(P_n^m) = \begin{cases} nk - \frac{n^2-m^2}{2m} & \text{if } \lceil \frac{n}{m} \rceil \text{ is odd and } m|n, \\ nk - \frac{n^2-s^2}{2m} + 1 & \text{if } \lceil \frac{n}{m} \rceil \text{ is odd and } m \nmid n, \\ nk - \frac{n^2}{2m} + 1 & \text{if } \lceil \frac{n}{m} \rceil \text{ is even and } m|n, \\ nk - \frac{n^2-(m-s)^2}{2m} + 1 & \text{if } \lceil \frac{n}{m} \rceil \text{ is even and } m \nmid n, \end{cases}$$

where $s \equiv n \pmod{m}$.

In this article, we develop a robust graph theoretic tool for the proof. Even though the tool is specifically used to prove our result, it can be adapted to prove bounds for other classes of graphs. Thus, we would like to remark that, the main contribution of this work is not only in proving an important result that captures a significant number of problems with a unified proof, but also in devising a proof technique that has the potential of becoming a standard technique to attack similar problems. We will prove the theorem in the next section.

Moreover, our proof of the upper bound is by giving a prescribed radio k-coloring of the concerned graph, and then proving its validity, while the lower bound proof establishes its optimality. Therefore, as a corollary to Theorem 1, we can say that our proof provides a linear time algorithm radio k-color powers of paths, optimally.

Theorem 2. *For all* $k > diam(P_n^m)$*, one can provide an optimal radio* k*-coloring of the graph* P_n^m *in* $O(n)$ *time.*

We prove Theorem 1 in the next section.

2 Proofs of Theorems 1 and 2

This section is entirely dedicated to the proofs of Theorems 1 and 2. The proof uses specific notations and terminologies developed for making it easier for the reader to follow. The proof is contained in several observations and lemmas and uses a modified and improved version of the DGNS formula [4] applicable for graphs having small diameters, that is, less than or equal to k.

As seen from the theorem statement, the graph P_n^m that we work on is the m^{th} power of the path on $(n+1)$ vertices. One crucial aspect of this proof is the naming of the vertices of P_n^m. In fact, for convenience, we shall assign *two* names to each of the vertices of the graph and use them as required depending on the context. Such a naming convention will depend on the parity of the diameter of P_m^n.

Observation 1. *The diameter of the graph* P_n^m *is* $diam(P_n^m) = \lceil \frac{n}{m} \rceil$.

For the rest of this section, we shall fix the notation that $q = \lfloor \frac{diam(P_n^m)}{2} \rfloor$.

2.1 The naming conventions

We are now ready to present the first naming convention for the vertices of P_n^m. For convenience, let us suppose that the vertices of P_n^m are placed (embedded) on the X-axis having co-ordinates $(i, 0)$ where $i \in \{0, 1, \cdots, n\}$ and two (distinct) vertices are adjacent if and only if their Euclidean distance is at most m.

We start by selecting the layer L_0 consisting of the vertex, named c_0, say, positioned at $(qm, 0)$ for even values of $diam(P_n^m)$. On the other hand, for odd values of $diam(P_n^m)$, the layer L_0 consists of the vertices $c_0, c_1, \cdots, c_m$, say, positioned at $(qm, 0), (qm+1, 0), \cdots, (qm+m, 0)$, respectively, and inducing a maximal clique of size $(m+1)$. The vertices of L_0 are called the *central vertices*, and those positioned to the left and the right side of the central vertices are naturally called the *left vertices* and the *right vertices*, respectively.

After this, we define the layer L_i as the set of vertices that are at a distance i from L_0. Observe that the layer L_i is non-empty for all $i \in \{0, 1, \cdots, q\}$. Moreover, notice that, for all $i \in \{1, 2, \cdots, q\}$, L_i consists of both left and right vertices. In particular, for $i \geq 1$, the left vertices of L_i are named $l_{i1}, l_{i2}, \cdots, l_{im}$, sorted according to the increasing order of their Euclidean distances from L_0. Similarly, for $i \in \{1, 2, \cdots, q-1\}$, the right vertices of L_i are named $r_{i1}, r_{i2}, \cdots$, r_{im}, sorted according to the increasing order of their Euclidean distance from L_0. However, the right vertices of L_q are $r_{q1}, r_{q2}, \cdots, r_{qs}$, where $s = (n+1) - (2q-1)m - |L_0|$ (observe that this s is the same as the s mentioned in the statement of Theorem 1), again sorted according to the increasing order of their Euclidean distances from L_0. That is, if $m \nmid n$, then there are $s = (n+1) - (2q-1)m - |L_0|$ right vertices in L_q. Besides, every layer L_i, for $i \in \{1, 2, \cdots, q-1\}$, has exactly m left vertices and m right vertices. This completes our first naming convention.

Now, we move to the second naming convention. This depends on yet another observation.

Observation 2. *Let ϕ be a radio k-coloring of P_n^m. Then $\phi(x) \neq \phi(y)$ for all distinct $x, y \in V(P_n^m)$.*

Proof. As $diam(P_n^m) < k$, the distance between any two vertices of P_n^m is at most $k-1$. Thus, their colors must differ by a value of at least 1. □

Let ϕ be a radio k-coloring of P_n^m. Thus, due to Observation 2, it is possible to sort the vertices of P_n^m according to the increasing order of their colors. That is, our second naming convention which names the vertices of P_n^m as $v_0, v_1, \cdots, v_n$ satisfying $\phi(v_0) < \phi(v_1) < \cdots < \phi(v_n)$. Clearly, the second naming convention depends only on the coloring ϕ, which, for the rest of this section, will play the role of any arbitrary radio k-coloring of P_n^m.

2.2 The lower bound

Next, we shall proceed to establish the lower bound of the Theorem 1 by showing it to be a lower bound of $span(\phi)$. To do so, however, we need to introduce yet another notation. Let $f : V(P_n^m) \to \{0, 1, \cdots, q\}$ be the function which indicates the layer of a vertex, that is, $f(x) = i$ if $x \in L_i$. With this notation, we initiate the lower bound proof with the following result.

Lemma 1. *For any $i \in \{0, 1, \cdots, n-1\}$, we have*

$$\phi(v_{i+1}) - \phi(v_i) \geq \begin{cases} k - f(v_i) - f(v_{i+1}) + 1 & \text{if } diam(P_n^m) \text{ is even,} \\ k - f(v_i) - f(v_{i+1}) & \text{if } diam(P_n^m) \text{ is odd.} \end{cases}$$

Proof. If $diam(P_n^m)$ is even, then L_0 consists of the single vertex c_0. Observe that, as v_i is in $L_{f(v_i)}$, it is at a distance $f(v_i)$ from c_0. Similarly, v_{i+1} is at a distance $f(v_{i+1})$ from c_0. Hence, by the triangle inequality, we have

$$d(v_i, v_{i+1}) \leq d(v_i, c_0) + d(c_0, v_{i+1}) = f(v_i) + f(v_{i+1}).$$

Therefore, by the definition of radio k-coloring,

$$\phi(v_{i+1}) - \phi(v_i) \geq k - f(v_i) - f(v_{i+1}) + 1.$$

If $diam(P_n^m)$ is odd, then L_0 is a clique. Thus, by the definition of layers and the function f, there exist vertices c_j and $c_{j'}$ in L_0 satisfying $d(v_i, c_j) = f(v_i)$ and $d(v_{i+1}, c_{j'}) = f(v_{i+1})$. Hence, by triangle inequality again, we have

$$d(v_i, v_{i+1}) \leq d(v_i, c_j) + d(c_j, c_{j'}) + d(c_{j'}, v_{i+1}) = f(v_i) + 1 + f(v_{i+1}).$$

Therefore, by the definition of radio k-coloring,

$$\phi(v_{i+1}) - \phi(v_i) \geq k - f(v_i) - f(v_{i+1}).$$

Hence we are done. □

Notice that it is not possible to improve the lower bound of the inequality presented in Lemma 1. Motivated by this fact, whenever we have

$$\phi(v_{i+1}) - \phi(v_i) = \begin{cases} k - f(v_i) - f(v_{i+1}) + 1 & \text{if } diam(P_n^m) \text{ is even,} \\ k - f(v_i) - f(v_{i+1}) & \text{if } diam(P_n^m) \text{ is odd.} \end{cases}$$

for some $i \in \{0, 1, \cdots, n-1\}$, we say that the pair (v_i, v_{i+1}) is *optimally colored* by ϕ. Moreover, we can naturally extend this definition to a sequence of vertices of the type $(v_i, v_{i+1}, \cdots, v_{i+i'})$ by calling it an *optimally colored sequence* by ϕ if (v_{i+j}, v_{i+j+1}) is optimally colored by ϕ for all $j \in \{0, 1, \cdots, i'-1\}$. Furthermore, a *loosely colored sequence* $(v_i, v_{i+1}, v_{i+2}, \cdots, v_{i+i'})$ is a sequence that does not contain any optimally colored sequence as a subsequence.

An important thing to notice is that the sequence of vertices $(v_0, v_1, \cdots, v_n)$ can be written as a concatenation of maximal optimally colored sequences and loosely colored sequences. That is, it is possible to write

$$(v_0, v_1, \cdots, v_n) = Y_0 X_1 Y_1 X_2 \cdots X_t Y_t$$

where Y_is are loosely colored sequences and X_js are maximal optimally colored sequences. Here, we allow the Y_is to be empty sequences as well. In fact, a Y_i is empty if and only if there exist two consecutive vertices $v_{s'}$ and $v_{s'+1}$ of P_n^m in the second naming convention such that $(v_{s'}, v_{s'+1})$ is loosely colored and that $X_i = (v_s, v_{s+1}, \cdots, v_{s'})$ and $X_{i+1} = (v_{s'+1}, v_{s'+2}, \cdots, v_{s''})$ for some $s \leq s' < s''$. By convention, empty sequences are always loosely colored and a sequence having a singleton vertex is always optimally colored. From now onward, whenever we mention a radio k-coloring ϕ of P_n^m, we shall also suppose an associated concatenated sequence using the same notation as mentioned above.

Let us now prove a result which plays an instrumental role in the proof of the lower bound.

Lemma 2. *Let ϕ be a radio-k coloring of P_n^m such that*

$$(v_0, v_1, \cdots, v_n) = Y_0 X_1 Y_1 X_2 \cdots X_t Y_t.$$

Then, for even values of $diam(P_n^m)$, we have

$$span(\phi) \geq \left[n(k+1) - 2\sum_{i=1}^{q} i|L_i| \right] + \left[f(v_0) + f(v_n) + \sum_{i=0}^{t} |Y_i| + t - 1 \right]$$

and, for odd values of $diam(P_n^m)$, we have

$$span(\phi) \geq \left[nk - 2\sum_{i=1}^{q} i|L_i| \right] + \left[f(v_0) + f(v_n) + \sum_{i=0}^{t} |Y_i| + t - 1 \right],$$

where $|Y_i|$ denotes the length of the sequence Y_i.

Proof. We know that $span(\phi) = \phi(v_n) - \phi(v_0)$. However, we can expand this difference as

$$\begin{aligned} span(\phi) &= \phi(v_n) - \phi(v_0) \\ &= (\phi(v_n) - \phi(v_{n-1})) + (\phi(v_{n-1}) - \phi(v_{n-2})) + \cdots + (\phi(v_1) - \phi(v_0)) \\ &= \sum_{i=0}^{n-1} [\phi(v_{i+1}) - \phi(v_i)]. \end{aligned}$$

Now, let ϵ be an indicator function on the diameter of P_n^m. That is, let $\epsilon = 1$ for even values of $diam(P_n^m)$ and $\epsilon = 0$ for odd values of $diam(P_n^m)$. Then, notice that, by Lemma 1, we have

$$\phi(v_{i+1}) - \phi(v_i) \geq k - f(v_i) - f(v_{i+1}) + \epsilon$$

and, if (v_i, v_{i+1}) is loosely colored, then

$$\phi(v_{i+1}) - \phi(v_i) > k - f(v_i) - f(v_{i+1}) + \epsilon.$$

Therefore, if

$$S = \{v_i : (v_i, v_{i+1}) \text{ is loosely colored, where } 0 \leq i \leq n-1\},$$

then we have,

$$\begin{aligned} span(\phi) &= \sum_{i=0}^{n-1} [\phi(v_{i+1}) - \phi(v_i)] \\ &\geq |S| + \sum_{i=0}^{n-1} [k - f(v_i) - f(v_{i+1}) + \epsilon] \\ &= |S| + n(k+\epsilon) - \sum_{i=0}^{n-1} f(v_i) - \sum_{i=0}^{n-1} f(v_{i+1}) \\ &= |S| + n(k+\epsilon) + f(v_0) + f(v_n) - 2\sum_{i=0}^{n} f(v_i) \\ &= |S| + n(k+\epsilon) + f(v_0) + f(v_n) - 2\sum_{i=0}^{q} i|L_i|. \end{aligned}$$

Notice that, to count $|S|$ it is enough to count the lengths of the loosely colored sequences, i.e. the $|Y_i|$s, and the number of transitions between the loosely colored and the optimally colored sequences, i.e. between a Y_i and an X_i. To be precise, we can write

$$\begin{aligned} |S| &= |Y_0| + (|Y_1| + 1) + (|Y_2| + 1) + \cdots + (|Y_{t-1}| + 1) + |Y_t| \\ &= (t-1) + \sum_{i=0}^{t} |Y_t|. \end{aligned}$$

Combining the above two equations therefore, we obtain the result. □

As we shall calculate the two additive components of Lemma 2 separately, we introduce short-hand notations for them for the convenience of reference. So, let

$$\alpha_1 = \begin{cases} n(k+1) - 2\sum_{i=1}^{q} i|L_i| & \text{if } diam(P_n^m) \text{ is even,} \\ nk - 2\sum_{i=1}^{q} i|L_i| & \text{if } diam(P_n^m) \text{ is odd,} \end{cases}$$

and

$$\alpha_2(\phi) = f(v_0) + f(v_n) + \sum_{i=0}^{t} |Y_i| + t - 1.$$

Observe that α_1 and α_2 are functions of a number of variables and factors such as, n, m, k, ϕ, etc. However, to avoid clumsy and lengthy formulations, we have avoided writing α_1 and α_2 as multivariate functions, as their definitions are not ambiguous in the current context. Furthermore, as k and P_n^m are assumed to be fixed in the current context and, as α_1 does not depend on ϕ (follows from its definition), it is treated and expressed as a constant as a whole. On the other hand, α_2 is expressed as a function of ϕ.

Now we shall establish lower bounds for α_1 and $\alpha_2(\phi)$ separately to prove the lower bound of Theorem 1. Let us start with α_1 first.

Lemma 3. *We have*

$$\alpha_1 = \begin{cases} nk - \frac{n^2+m^2-s^2}{2m} & \text{if } diam(P_n^m) \text{ is even,} \\ nk - \frac{n^2-s^2}{2m} & \text{if } diam(P_n^m) \text{ is odd,} \end{cases}$$

where $s = (n+1) - (2q-1)m - |L_0|$.

Proof. Notice that $|L_i| = 2m$ for all $i \in \{1, 2, \cdots, q-1\}$ and $|L_q| = m + s$. So, simply replacing these values in the definition of α_1 and using the relation $s = n - (2q - 1 + \epsilon)m$, where $\epsilon = 0$ for even values of $diam(P_n^m)$ and $\epsilon = 1$ for odd values of $diam(P_n^m)$, gives us the result. □

Next, we focus on $\alpha_2(\phi)$. We shall handle the cases with odd $diam(P_n^m)$ first.

Lemma 4. *We have*

$$\alpha_2(\phi) \geq \begin{cases} 0 & \text{if } diam(P_n^m) \text{ is odd and } m|n, \\ 1 & \text{if } diam(P_n^m) \text{ is odd and } m \nmid n. \end{cases}$$

Proof. First of all, notice that there is nothing to prove when $diam(P_n^m)$ is odd and $m|n$ as $\alpha_2(\phi)$ is always non-negative by definition. However, when $diam(P_n^m)$ is odd and $m \nmid n$, it is enough to show that $\alpha_2(\phi) \neq 0$. Suppose the contrary, that is, $\alpha_2(\phi) = 0$. Then, we must have both $f(v_0) = f(v_n) = 0$ and $(v_0, v_1, \cdots, v_n) = Y_0X_1Y_1$ having $Y_0 = Y_1 = \emptyset$. That is, both v_0 and v_n must be from L_0 and the whole sequence $(v_0, v_1, \cdots, v_n)$ must be an optimally colored sequence.

Observe that if l_{i1}, for any $i \in \{1, 2, \cdots, q\}$, is an element of an optimally colored pair, then the other element must be either c_m or r_{jm} for some $j \in \{1, 2, \cdots, q-1\}$. This follows from the distance constraints and the definition of an optimally colored pair of vertices. On the other hand, a pair of vertices in which one is c_m and the other is a right vertex is not an optimally colored pair of vertices. Moreover, any pair of left vertices $(l_{ia}, l_{i'a'})$ or any pair of right vertices $(r_{jb}, r_{j'b'})$ are also loosely colored each.

Thus, X_1 must contain a contiguous subsequence of the form $(a_1, b_1, a_2, b_2, \cdots, a_q, b_q)$ where a_is (resp., b_js) are from $\{l_{11}, l_{21}, \cdots, l_{q1}\}$ and b_js (resp., a_is) are from $\{c_m, r_{1m}, r_{2m}, \cdots, r_{(q-1)m}\}$.

If $a_1 \in \{l_{11}, l_{21}, \cdots, l_{q1}\}$, then $a_1 \neq v_0$, as $f(v_0) = 0 \neq f(a_1)$. Thus $a_1 = v_i$ for some $i \geq 1$. This is not possible as v_{i-1} cannot be from the set $\{c_m, r_{1m}, r_{2m}, \cdots, r_{(q-1)m}\}$ and therefore, the pair (v_{i-1}, v_i) is not optimally colored, a contradiction. Hence, $\alpha_2(\phi) \neq 0$.

Similarly, we can arrive at a contradiction if $b_q \in \{l_{11}, l_{21}, \cdots, l_{q1}\}$ and so, $\alpha_2(\phi) \neq 0$ in this case as well. Hence, we are done. □

Next, we consider the cases with even $diam(P_n^m)$. Before stating with it though, we are going to introduce some terminologies to be used during the proofs. So, let X_i be an optimally colored sequence. As X_i cannot have two consecutive left (resp., right) vertices as elements, the number of left vertices can be at most one more than the number of right vertices and the central vertex, the latter two combined together. Based on this observation, if the number of left vertices is more, equal, or less than the number of right vertices and the central vertex combined in X_i, then X_i is called a *leftist*, *balanced*, or *rightist* sequence, respectively.

Lemma 5. *We have*

$$\alpha_2(\phi) \geq \begin{cases} 1 & \textit{if } diam(P_n^m) \textit{ is even and } m|n, \\ m-s+1 & \textit{if } diam(P_n^m) \textit{ is even and } m \nmid n, \end{cases}$$

where $s \equiv n \pmod{m}$.

Proof. For even values of $diam(P_n^m)$, L_0 consists of only the vertex c_0. Therefore, at most one of v_0 and v_n can be equal to c_0 implying $f(v_0) + f(v_n) \geq 1$. This proves the case when $m|n$. So, let us now focus on the case when $m \nmid n$.

We know that there are exactly $(q-1)m + s$ right vertices and one central vertex c_0. Suppose that at most $(q-1)m+s$ vertices among the set of right and central vertices are part of optimally colored sequences of $(v_0, v_1, \cdots, v_n)$. Thus, the total number of vertices across the t optimally colored sequences will be

$$\sum_{i=1}^{t} |X_i| \leq 2(q-1)m + 2s + t.$$

That leaves us with

$$\sum_{i=0}^{t} |Y_i| \geq [(2q-1)m + s + 1] - [2(q-1)m + 2s + t] = m - s + 1 - t.$$

Recall that $f(v_0) + f(v_n) \geq 1$. Hence,

$$\alpha_2(\phi) = f(v_0) + f(v_n) + \sum_{i=0}^{t} |Y_i| + t - 1 \geq m - s + 1.$$

Therefore, we are left with the case when all $(q-1)m + s + 1$ right and central vertices are part of optimally colored sequences of $(v_0, v_1, \cdots, v_n)$. Suppose that the number of leftist, balanced, and rightist sequences are t_1, t_2, and t_3, respectively, where $t_1 + t_2 + t_3 = t$. In this case

$$\sum_{i=1}^{t} |X_i| \leq 2(q-1)m + 2s + 2 + t_1 - t_3.$$

That leaves us with

$$\sum_{i=0}^{t} |Y_i| \geq [(2q-1)m+s+1]-[2(q-1)m+2s+2+t_1-t_3] = (m-s-1)-(t_1-t_3).$$

Hence,

$$\alpha_2(\phi) \geq f(v_0)+f(v_n)+\sum_{i=0}^{t} |Y_i|+t-1 \geq (m-s-2)+[f(v_0)+f(v_n)+t-t_1+t_3].$$

Thus, it is enough to show that

$$[f(v_0)+f(v_n)+t-t_1+t_3] \geq 3. \tag{1}$$

As $f(v_0)+f(v_n) \geq 1$, the equation (1) will be satisfied if there is one rightist sequence, or two balanced sequences. Furthermore, if $f(v_0)+f(v_n) \geq 2$, then equation (1) will be satisfied if there is one rightist or balanced sequence.

Notice that, if r_{i1}, for any $i \in \{1,2,\cdots,q\}$, is an element of an optimally colored pair, then the other element must be either c_0 or l_{jm} for some $j \in \{1,2,\cdots,q\}$. We know that all right vertices, in particular, $r_{11}, r_{21}, \cdots, r_{q1}$, are part of some optimally colored sequences. Observe that, if they are distributed over two or more optimally colored sequences, then due to the above property, either one of those sequences will be rightist, or two of the sequences will be balanced.

Moreover, if they are part of one optimally colored sequence X_i, then that sequence cannot be leftist. Furthermore, if the first or the last vertex of X_i is c_0, then X_i is rightist. Thus, in any case, equation (1) is satisfied. Hence we are done. □

Combining Lemmas 2, 3, 5 and 4, therefore, we have the following lower bound for the parameter $rc_k(P_n^m)$.

Lemma 6. *For all $k \geq diam(P_n^m)$ we have*

$$rc_k(P_n^m) \geq \begin{cases} nk - \frac{n^2-m^2}{2m} & \text{if } \lceil \frac{n}{m} \rceil \text{ is odd and } m|n, \\ nk - \frac{n^2-s^2}{2m} + 1 & \text{if } \lceil \frac{n}{m} \rceil \text{ is odd and } m \nmid n, \\ nk - \frac{n^2}{2m} + 1 & \text{if } \lceil \frac{n}{m} \rceil \text{ is even and } m|n, \\ nk - \frac{n^2-(m-s)^2}{2m} + 1 & \text{if } \lceil \frac{n}{m} \rceil \text{ is even and } m \nmid n, \end{cases}$$

where $s \equiv n \pmod{m}$.

2.3 The upper bound

Now let us prove the upper bound. We shall provide a radio k-coloring ψ of P_n^m and show that its span is the same as the value of $rc_k(P_n^m)$ stated in Theorem 1. To define ψ, we shall use both the naming conventions. That is, we shall express the ordering $(v_0, v_1, \cdots, v_n)$ of the vertices of P_n^m with respect to ψ in terms of the first naming convention.

Let us define a few ordering for the right (and similarly for the left) vertices:

(1) $r_{ij} \prec_1 r_{i'j'}$ if either (i) $j < j'$ or (ii) $j = j'$ and $(-1)^{j-1}i < (-1)^{j'-1}i'$;
(2) $r_{ij} \prec_2 r_{i'j'}$ if either (i) $j < j'$ or (ii) $j = j'$ and $(-1)^{m-j}i < (-1)^{m-j'}i'$;
(3) $r_{ij} \prec_3 r_{i'j'}$ if either (i) $j < j'$ or (ii) $j = j'$ and $i > i'$; and
(4) $r_{ij} \prec_4 r_{i'j'}$ if either (i) $j < j'$ or (ii) $j = j'$ and $(-1)^{j}i < (-1)^{j'}i'$.

Observe that, the orderings are based on comparing the second co-ordinate of the indices of the right (resp., left) vertices, and if they happen to be equal, then comparing the first co-ordinate of the indices with conditions on their parities. Moreover, all the above four orderings defines total orders on the set of all right (resp., left) vertices. Thus, there is a unique increasing (resp., decreasing) sequence of right (or the left) vertices with respect to $\prec_1$, $\prec_2$, $\prec_3$, and $\prec_4$. Based on these orderings, we are going to construct a sequence of vertices of the graph and then greedy color the vertices to provide our labeling.

The sequences of the vertices are given as follows:

(1) An *alternating chain* as a sequence of vertices of the form $(a_1, b_1, a_2, b_2, \cdots, a_p, b_p)$ such that $(a_1, a_2, \cdots, a_p)$ is the increasing sequence of right vertices with respect to $\prec_1$ and $(b_1, b_2, \cdots, b_p)$ is the decreasing sequence of left vertices with respect to $\prec_2$.
(2) A *reverse alternating chain* as a sequence of vertices of the form $(a_1, b_1, a_2, b_2, \cdots, a_p, b_p)$ such that $(a_1, a_2, \cdots, a_p)$ is the increasing sequence of left vertices with respect to $\prec_1$ and $(b_1, b_2, \cdots, b_p)$ is the decreasing sequence of right vertices with respect to $\prec_2$;
(3) A *canonical chain* as a sequence of vertices of the form $(a_1, b_1, a_2, b_2, \cdots, a_p, b_p)$ such that $(a_1, a_2, \cdots, a_p)$ is the increasing sequence of right vertices with respect to $\prec_3$ and $(b_1, b_2, \cdots, b_p)$ is the decreasing sequence of left vertices with respect to $\prec_3$;
(4) A *special alternating chain* as a sequence of vertices of the form $(a_1, b_1, a_2, b_2, \cdots, a_p, b_p)$ such that $(a_1, a_2, \cdots, a_p)$ is the increasing sequence of right vertices with respect to $\prec_2$ and $(b_1, b_2, \cdots, b_p)$ is the decreasing sequence of left vertices with respect to $\prec_1$; and
(5) A *special canonical chain* as a sequence of vertices of the form $(a_1, b_1, a_2, b_2, \cdots, a_p, b_p)$ such that $(a_1, a_2, \cdots, a_p)$ is the increasing sequence of right vertices with respect to $\prec_4$ and $(b_1, b_2, \cdots, b_p)$ is the decreasing sequence of left vertices with respect to $\prec_4$.

Notice that the special alternating chains, the reverse alternating chain and the canonical chains can exist only when the number of right and left vertices are equal. Of course, when $m|n$, both the chains exist. Otherwise, we shall modify the names of the vertices a little to make them exist.

We are now ready to express the sequence $(v_0, v_1, \cdots, v_n)$ by splitting it into different cases which are depicted in Figures 1, 2, 3, 4, 5 and 6 for example. In the figures, the both naming conventions for each of the vertices are given so that the reader may cross verify the correctness for that particular instance for each case. For convenience, also recall that $q = \lfloor \frac{diam(P_n^m)}{2} \rfloor$.

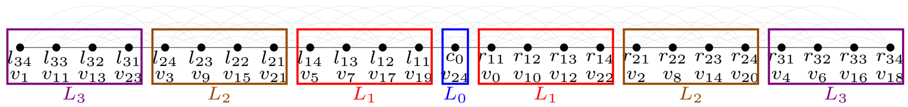

Fig. 1: *Case 1.* $n = 24$, $m = 4$, $diam(P^4_{24}) = 6$, $k = 7$.

Case 1: when $diam(P^m_n)$ is even, $m|n$ and $k > diam(P^m_n)$. First of all, $(v_0, v_1, \cdots, v_{2qm-1})$ is the alternating chain. Moreover, $v_n = c_0$.

Case 2: when $diam(P^m_n)$ is odd, $m|n$ and $k > diam(P^m_n)+1$. First of all, $v_0 = c_m$ and $(v_1, v_2, \cdots, v_{2qm})$ is the reverse alternating chain. Moreover,

$$(v_{2qm+1}, v_{2qm+2}, \cdots, v_{2qm+m}) = (c_0, c_1, \cdots, c_{m-1}).$$

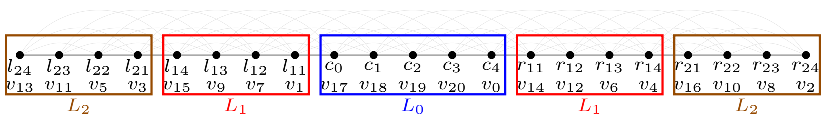

Fig. 2: *Case 2.* $n = 20$, $m = 4$, $diam(P^4_{20}) = 5$, $k = 7$.

Case 3: when $diam(P^m_n)$ is odd, $m \nmid n$ and $k > diam(P^m_n) + 1$. Notice that, in this case, the left vertices are $(m - s)$ more than the right vertices. Also, L_0 has $(m + 1)$ vertices in this case. We shall rename some of the vertices from L_0 and temporarily call them right vertices to compensate for the $(m - s)$ missing right vertices, and then present the ordering. To be specific, we assign the new names $c_i = r_{oi}$ for $i \in \{s+1, s+2, \cdots, m\}$. Counting the newly named central vertices as right vertices, we have an equal number of left and right vertices now. First of all, $(v_0, v_1, \cdots, v_{2qm-1})$ is the reverse alternating chain. Additionally,

$$(v_{2qm}, v_{2qm+1}, \cdots, v_{2qm+s}) = (c_0, c_1, \cdots, c_s).$$

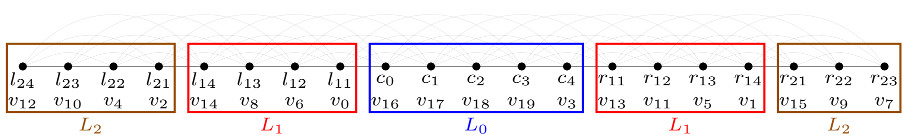

Fig. 3: *Case 3.* $n = 19$, $m = 4$, $diam(P^4_{19}) = 5$, $k = 7$, $s = 3$.

Case 4: when $diam(P^m_n)$ is even, $m \nmid n$ and $k > diam(P^m_n)$. Notice that, in this case, the left vertices are $(m - s)$ more than the right vertices. Also, L_0 has only one vertex in this case. We shall discard some vertices from the set of left vertices, and then present the ordering. To be specific, we disregard the subset $\{l_{11}, l_{12}, \cdots, l_{1(m-s)}\}$, temporarily, from the set of left vertices and consider the alternating chain. First of all, $(v_0, v_1, \cdots, v_{2qm-2m+2s-1})$ is the alternating chain. Additionally, $(v_{2qm-2m+2s}, v_{2qm-2m+2s+1}, v_{2qm-2m+2s+2}, \cdots, v_{2qm-m+s}) = (c_0, l_{11}, l_{12}, \cdots, l_{1(m-s)})$.

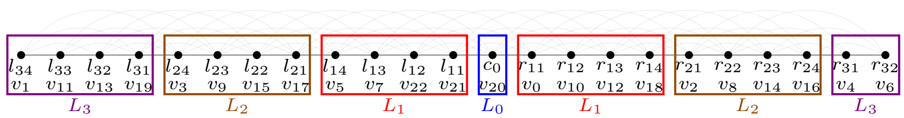

Fig. 4: *Case 4.* $n = 22$, $m = 4$, $diam(P^4_{22}) = 6$, $k = 7$, $s = 2$.

Case 5: when $diam(P^m_n)$ is odd, $m|n$ and $k = diam(P^m_n) + 1$. Let the ordering of the vertices be $(v_0, v_1, \cdots, v_{2qm+m})$. Now, $v_{j(2q+1)} = c_j$ for all $0 \leq j \leq m$. The remaining vertices follow the canonical chain.

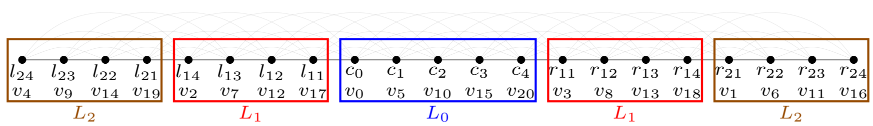

Fig. 5: *Case 5.* $n = 20$, $m = 4$, $diam(P^4_{20}) = 5$, $k = 6$.

Case 6: when $diam(P^m_n)$ is odd, $m \nmid n$ and $k = diam(P^m_n) + 1$. For any set A, let $A^\star$ represent an ordered sequence of the elements of A. Let $G \cong P^m_n$ and $S = V(G) = \{v_0, v_1, v_2, \cdots, v_{2qm+s}\}$. Then $S^\star$ is defined as described. First, define

$$T = \{v_t : 0 \leq t \leq s(2q+1)\} - \{v_{j(2q+1)} : 0 \leq j \leq s\}.$$

Order $T^\star$ as canonical chain. Also, define $v_{j(2q+1)} = c_j$ for all $0 \leq j \leq s$. Assume G' to be the subgraph of G induced by the subset $S - \{r_{q1}, r_{q2}, \cdots, r_{qs}\}$ of S. Then $G' \cong P^m_{n'}$, $m|n'$ and $diam(G) = \frac{n'}{m}$ is even, where $n' = n - s$. Define

$$v_n = l_{11} \text{ and } U = \{v_t : s(2q+1) + 1 \leq t < n\}.$$

Note that $U \subset V(G')$. Order $U^\star$ (as vertices of G') by the following.

(i) Special alternating chain when m and s have the same parity.
(ii) Alternating chain when m is even and s is odd.
(iii) Special canonical chain when m is odd and s is even.

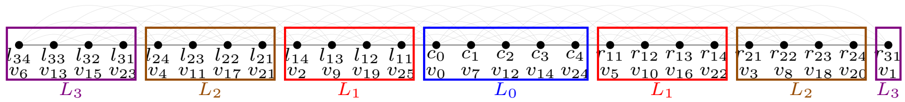

Fig. 6: *Case 6.* $n = 25$, $m = 4$, $diam(P_{25}^4) = 7$, $k = 8$, $s = 1$.

Thus, we have obtained a sequence $(v_0, v_1, \cdots, v_n)$ in each case under consideration. Now, we define, $\psi(v_0) = 0$ and $\psi(v_{i+1}) = \psi(v_i) + k + 1 - d(v_i, v_{i+1})$, recursively, for all $i \in \{1, 2, \cdots, n-1\}$. Next, we note that ψ is a radio k-coloring.

Lemma 7. *The function ψ is a radio k-coloring of P_n^m.*

Proof. Notice that, the way ψ is defined, for all $i \in \{0, 1, \cdots, n-1\}$, we have $\psi(v_{i+1}) - \psi(v_i) = k + 1 - d(v_i, v_{i+1})$. Furthermore, one can observe that for all $i \in \{0, 1, \cdots, n-2\}$, we have $\psi(v_{i+2}) - \psi(v_i) \geq k$. As the value of the image of ψ increases with respect to the indices of v_is, ψ satisfies the conditions for being a radio k-coloring. □

This brings us to the upper bound for $rc_k(P_n^m)$.

Lemma 8. *For all $k > diam(P_n^m)$, we have*

$$rc_k(P_n^m) \leq \begin{cases} nk - \frac{n^2-m^2}{2m} & \text{if } \lceil \frac{n}{m} \rceil \text{ is odd and } m|n, \\ nk - \frac{n^2-s^2}{2m} + 1 & \text{if } \lceil \frac{n}{m} \rceil \text{ is odd and } m \nmid n, \\ nk - \frac{n^2}{2m} + 1 & \text{if } \lceil \frac{n}{m} \rceil \text{ is even and } m|n, \\ nk - \frac{n^2-(m-s)^2}{2m} + 1 & \text{if } \lceil \frac{n}{m} \rceil \text{ is even and } m \nmid n, \end{cases}$$

where $s \equiv n \pmod m$.

Proof. Observe that, $rc_k(P_n^m) \leq span(\psi)$. So, to prove the upper bound, it is enough to show that for all $k > diam(P_n^m)$ and $s \equiv n \pmod m$,

$$span(\psi) = \begin{cases} nk - \frac{n^2-m^2}{2m} & \text{if } \lceil \frac{n}{m} \rceil \text{ is odd and } m|n, \\ nk - \frac{n^2-s^2}{2m} + 1 & \text{if } \lceil \frac{n}{m} \rceil \text{ is odd and } m \nmid n, \\ nk - \frac{n^2}{2m} + 1 & \text{if } \lceil \frac{n}{m} \rceil \text{ is even and } m|n, \\ nk - \frac{n^2-(m-s)^2}{2m} + 1 & \text{if } \lceil \frac{n}{m} \rceil \text{ is even and } m \nmid n. \end{cases}$$

Notice that, for odd values of $diam(P_n^m)$ and for even values of $diam(P_n^m)$ where $m|n$, the whole sequence $(v_0, v_1, \cdots, v_n)$ is optimally colored with respect to ψ. Moreover, note that

$$f(v_0) + f(v_n) = \begin{cases} 0 & \text{if } \lceil \frac{n}{m} \rceil \text{ is odd and } m|n, \\ 1 & \text{if } \lceil \frac{n}{m} \rceil \text{ is odd and } m \nmid n, \\ 1 & \text{if } \lceil \frac{n}{m} \rceil \text{ is even and } m|n. \end{cases}$$

Thus, adding these values with α_1 (from Lemma 3) will complete the proof for the first three cases.

For the final case, that is, for even values of $diam(P_n^m)$ where $m \nmid n$, the sequence $(v_0, v_1, \cdots, v_{2(q-1)m+2s+1})$ is an optimally colored sequence. On the other hand,
$(v_{2(q-1)m+2s+2}, v_{2(q-1)m+2s+3}, \cdots, v_n)$ is a loosely colored sequence. Thus, the whole sequence has exactly $(m-s-1)$ loosely colored pairs, namely,
$(v_{2(q-1)m+2s+1}, v_{2(q-1)m+2s+2})$, $(v_{2(q-1)m+2s+2}, v_{2(q-1)m+2s+3})$, $\cdots$, (v_{n-1}, v_n). These pairs are nothing but (l_{11}, l_{12}), (l_{12}, l_{11}), $\cdots$, $(l_{1(m-s-1)}, l_{1(m-s)})$. Now, let us count how many extra colors are skipped for each pair. In fact, we claim that the number of extra colors skipped for the pair $(l_{1i}, l_{1(i+1)})$ is one, for all $i \in \{1, 2, \cdots, m-s-1\}$. Notice that, both l_{1i} and $l_{1(i+1)}$ are from L_1. Thus, if they were optimally colored, we would have had

$$\psi(l_{1(i+1)}) = \psi(l_{1i}) + k + 1 - f(l_{1(i+1)}) - f(l_{1i}) = \psi(l_{1i}) + k - 1.$$

However, the distance between l_{1i} and $l_{1(i+1)}$ is one. Thus, what we actually have is

$$\psi(l_{1(i+1)}) = \psi(l_{1i}) + k + 1 - d(l_{1(i+1)}, l_{1i}) = \psi(l_{1i}) + k.$$

Thus, a total of extra $(m-s-1)$ colors are skipped while coloring the said loosely colored sequence. Moreover, as $f(v_0) + f(v_n) = 2$ in this case, we have $span(\psi) = \alpha_1 + (m-s-1) + 2$. Hence, simply replacing the value of α_1 from Lemma 3 in the above equation ends the proof. □

2.4 The proofs

Finally we are ready to conclude the proofs.

Proof of Theorem 1 The proof follows directly from the Lemmas 6 and 8. □

Proof of Theorem 2 Notice that the proof of the upper bound for Theorem 1 is given by prescribing an algorithm (implicitly). The algorithm requires ordering the vertices of the input graph, and then providing the coloring based on the ordering. Each step runs in linear order of the number of vertices in the input graph. Moreover, we have theoretically proved the tightness of the upper bound. Thus, we are done. □

Acknowledgements: This work is partially supported by the following projects: "MA/IFCAM/18/39", "SRG/2020/001575", "MTR/2021/000858", and "NBHM/RP-8 (2020)/Fresh". Research by the first author is partially sponsored by a public grant overseen by the French National Research Agency as part of the "Investissements d'Avenir" through the IMobS3 Laboratory of Excellence (ANR-10-LABX-0016) and the IDEX-ISITE initiative CAP 20-25 (ANR-16-IDEX-0001).

References

1. G. Chartrand, D. Erwin, F. Harary, and P. Zhang. Radio labelings of graphs. *Bulletin of the Institute of Combinatorics and its Applications*, 33:77–85, 2001.
2. G. Chartrand, D. Erwin, and P. Zhang. A graph labeling problem suggested by FM channel restrictions. *Bulletin of the Institute of Combinatorics and its Applications*, 43:43–57, 2005.
3. G. Chartrand, L. Nebeskỳ, and P. Zhang. Radio k-colorings of paths. *Discussiones Mathematicae Graph Theory*, 24(1):5–21, 2004.
4. S. Das, S. C. Ghosh, S. Nandi, and S. Sen. A lower bound technique for radio k-coloring. *Discrete Mathematics*, 340(5):855–861, 2017.
5. J. A. Gallian. A dynamic survey of graph labeling. *Electronic Journal of combinatorics*, 1(DynamicSurveys):DS6, 2018.
6. J. R. Griggs and R. K. Yeh. Labelling graphs with a condition at distance 2. *SIAM Journal on Discrete Mathematics*, 5(4):586–595, 1992.
7. F. Havet, B. Reed, and J.-S. Sereni. L (2, 1)-labelling of graphs. In *ACM-SIAM symposium on Discrete algorithms (SODA 2008)*, pages 621–630, 2008.
8. M. Kchikech, R. Khennoufa, and O. Togni. Linear and cyclic radio k-labelings of trees. *Discussiones Mathematicae Graph Theory*, 27(1):105–123, 2007.
9. R. Khennoufa and O. Togni. A note on radio antipodal colourings of paths. *Mathematica Bohemica*, 130(3):277–282, 2005.
10. S. R. Kola and P. Panigrahi. Nearly antipodal chromatic number $ac'(P_n)$ of the path P_n. *Mathematica Bohemica*, 134(1):77–86, 2009.
11. D. D. F. Liu and M. Xie. Radio number for square paths. *Ars Combinatoria*, 90:307–319, 2009.
12. D. D. F. Liu and X. Zhu. Multilevel distance labelings for paths and cycles. *SIAM Journal on Discrete Mathematics*, 19(3):610–621, 2005.
13. P. Panigrahi. A survey on radio k-colorings of graphs. *AKCE International Journal of Graphs and Combinatorics*, 6(1):161–169, 2009.
14. D. B. West. *Introduction to Graph Theory (2^{nd} Edition).* Prentice Hall, 2001.