On radio k-coloring of the power of the infinite path

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Abstract

The radio k-chromatic number $rc_k(G)$ of a graph G is the minimum integer ℓ such that there is a mapping f from the vertices of G to the set of integers $\{0, 1, \ldots, \ell\}$ satisfying $|f(u) - f(v)| \geq k + 1 - d(u, v)$ for any two distinct vertices $u, v \in V(G)$, where $d(u, v)$ denotes the distance between u and v. To date, the radio k-chromatic number of finite paths and square of finite paths is computed exactly when k is equal to the diameter of the graph. Moreover, the exact value of the radio k-chromatic number of an arbitrary power of a finite path is also computed when the diameter of the path is strictly smaller than k. Close lower and upper bounds for the radio k-chromatic number are given for the infinite path. Furthermore, lower and upper bounds have been found for an arbitrary power greater than or equal to two of infinite path. In this article, we improve the lower bound for $rc_k(G)$ of any arbitrary power of the infinite path for any k. In particular, when k is even, we halve the gap between the known lower and upper bounds.

Keywords: radio k-coloring, radio k-chromatic number, infinite path, power of paths.

1 Introduction and the main theorem

The *Channel Assignment Problem* (CAP) addresses the issue of optimally assigning frequencies to the nodes in a wireless network to avoid interference in communication. Loosely speaking, efficient solution in this front leads to an improvement in the network speed and contributes to upgrades from 2G to 5G networks.

One of the most popular graph theoretic models of this problem is defined as the radio coloring. An *ℓ-radio k-coloring* of a graph G is a function $f : V(G) \rightarrow \{0, 1, \ldots, \ell\}$

satisfying

$$|f(u) - f(v)| \geq k + 1 - d(u, v)$$

for any two distinct vertices $u, v \in V(G)$, where $d(u, v)$ denotes the distance between u and v. The *radio k-chromatic number* $rc_k(G)$ of G is the minimum ℓ such that G admits an ℓ-radio k-coloring.

The most famous research in this topic involves the special case of radio 2-coloring, better known as the $L(2, 1)$-labeling. In that, we have the celebrated Griggs and Yeh conjecture [4] which claims $rc_2(G) \leq \Delta^2$, where Δ is the maximum degree of G. To date, Havet, Reed, and Sereni [5] have shown the conjecture to be true for all $\Delta \geq 10^{69}$.

While evaluating the radio k-chromatic number in general for different graph families, a particular focus has been given on evaluating it for paths and their powers. Notable work on this front includes the result due to Liu and Zhu [8] where they managed to compute the exact value of the radio k-chromatic number of paths when k is equal to the diameter of the graph.

Theorem 1.1. *[8] Let P_k be a path on k edges. For any $k \geq 3$,*

$$rc_k(P_k) = \begin{cases} \dfrac{k^2 + 4}{2} & \text{if } k \text{ is even,} \\ \dfrac{k^2 + 1}{2} & \text{if } k \text{ is odd.} \end{cases}$$

Furthermore, the lower and the upper bounds of the radio k-chromatic number of the infinite paths are also studied. However, the best known lower and the upper bounds are due to two different bodies of works [3, 6].

Theorem 1.2. *[3, 6] Let P be the infinite path. Then,*

$$\frac{k^2 + k}{2} \leq rc_k(P) \leq \begin{cases} \dfrac{k^2 + 2k}{2} & \text{if } k \text{ is even,} \\ \dfrac{k^2 + 2k - 1}{2} & \text{if } k \text{ is odd.} \end{cases}$$

Notably, the upper bound of the above result is conjectured to be tight [6]. Also, the exact value of the radio k-chromatic number has been proved for square of paths [7] when k is equal to the diameter of the graph.

Theorem 1.3. *[7] Let P_n be the path on n edges. Let $k = diam(P_n^2)$. For any $n \geq 2$,*

$$rc_k(P_n^2) = \begin{cases} k^2 + 2 & \text{if } n \equiv 0 \pmod 4 \text{ and } n \geq 8, \\ k^2 + 1 & \text{otherwise.} \end{cases}$$

Recently, the exact value of the radio k-chromatic number of the powers of paths is proved for graphs having small diameter [2].

Theorem 1.4. *[2] Let P_n^m denote the m^{th} power of the path P_n on n edges. For all $k > diam(P_n^m)$ for even values of $diam(P_n^m)$ and $k > diam(P_n^m) + 1$ for odd values of $diam(P_n^m)$,*

$$rc_k(P_n^m) = \begin{cases} nk - \dfrac{n^2 - m^2}{2m} & \text{if } \lceil \frac{n}{m} \rceil \text{ is odd and } m \mid n, \\ nk - \dfrac{n^2 - s^2}{2m} + 1 & \text{if } \lceil \frac{n}{m} \rceil \text{ is odd and } m \nmid n, \\ nk - \dfrac{n^2}{2m} + 1 & \text{if } \lceil \frac{n}{m} \rceil \text{ is even and } m \mid n, \\ nk - \dfrac{n^2 - (m+s)^2}{2m} + 1 & \text{if } \lceil \frac{n}{m} \rceil \text{ is even and } m \nmid n, \end{cases}$$

where $s \equiv n \pmod{m}$.

A lower bound and an upper bound for the radio k-chromatic number of powers of infinite path have been found in [1].

Theorem 1.5. *[1] Let P^m denote the m^{th} power of the infinite path. Then for $m \geq 2$,*

$$\frac{mk^2 + 1}{2} \leq rc_k(P^m) \leq \begin{cases} \dfrac{mk^2 + mk}{2} & \text{if } k \text{ is odd,} \\ \dfrac{mk^2 + 2k}{2} & \text{if } k \text{ is even.} \end{cases}$$

We improve the lower bound of the radio k-chromatic number of powers of infinite path by a linear factor of k, for both odd and even values of k. To be precise, given a graph G, its m^{th} power is the graph G^m obtained by adding edges between all vertices which are at distance at most m from each other.

Theorem 1.6. *Let P^m denote the m^{th} power of the infinite path. Then we have the following lower bound for $m \geq 2$.*

$$rc_k(P^m) \geq \begin{cases} \dfrac{mk^2 + m + k - 1}{2} & \text{if } k \text{ is odd,} \\ \dfrac{mk^2 + k}{2} & \text{if } k \text{ is even.} \end{cases}$$

In this article, we follow the standard graph theoretic notations as per West [9] and all the graphs considered are simple. Also, we prove the above result in the Section 2.

2 Proof of Theorem 1.6

Given a graph G and a set S of vertices, $N(S)$ denotes the set of vertices which are adjacent to at least one vertex in S.

Let us assume that the vertices of P^m are placed on the point $(j, 0)$ (where j is an integer) on the x-axis and two such vertices are adjacent if and only if their Euclidean distance is at most m. Moreover, call the vertex positioned at $(j, 0)$ as v_j.

Let $k = 2p$ when k is even and $k = 2p + 1$ when k is odd. We start by defining a vertex subset

$$L_0 = \begin{cases} \{v_0, v_1, \ldots, v_m\} & \text{if } k \text{ is odd,} \\ \{v_0\} & \text{if } k \text{ is even.} \end{cases}$$

After that we recursively define the sets

$$L_{i+1} = N(L_i) \setminus \bigcup_{j=0}^{i} L_j$$

for all $i \in \{0, 1, \ldots, p-1\}$. Then, we define the set

$$D_k = \bigcup_{j=0}^{p} L_j.$$

Notice that the diameter of the induced graph $P^m[D_k]$ is at most k, and thus under any radio k-coloring f of P^m, the colors assigned to the vertices of D_k must be distinct.

Finally, let $x, y \in D_k$ be the vertices for which $f(x), f(y)$ attain the maximum and the minimum values, respectively, when x, y vary over D_k. Moreover, assume that α and β are such that we have $x \in L_\alpha$ and $y \in L_\beta$. These α and β are unique as the vertices of D_k get distinct colors. Without loss of generality, we may assume that $\alpha = p$ (as otherwise, we may translate our initial choice of L_0 somewhere else to force this).

Hence, according to the DGNS formula proved in [3],

$$rc_k(P^m) \geq (|D_k| - 2p - 1) + (\alpha + \beta) + 2\sum_{i=0}^{p} |L_i|(p-i).$$

Thus it is possible to calculate the lower bound if we can find out the cardinalities of L_i for all $i \in \{0, 1, \cdots, p\}$.

Note that, the cardinality

$$|L_0| = \begin{cases} m+1 & \text{if } k \text{ is odd,} \\ 1 & \text{if } k \text{ is even.} \end{cases}$$

by our choice. Furthermore, one can observe that $|L_i| = 2m$ for all $i \in \{1, 2, \cdots, p\}$. Therefore,

$$|D_k| = \sum_{i=0}^{p} |L_i| = \begin{cases} 2pm + m + 1 & \text{if } k \text{ is odd,} \\ 2pm + 1 & \text{if } k \text{ is even.} \end{cases}$$

Hence, for odd values of k, we have

$$\begin{aligned} rc_k(P^m) &\geq (|D_k| - 2p - 1) + (\alpha + \beta) + 2\sum_{i=0}^{p} |L_i|(p-i) \\ &\geq (2pm + m + 1 - 2p - 1) + p + 2\sum_{i=0}^{p} |L_i|(p-i) \\ &= 2pm + m - p + 2(m+1)p + 4m\sum_{i=1}^{p}(p-i) \\ &= 4pm + m + p + 4m\left(\frac{p(p-1)}{2}\right) \\ &= m(2p^2 + 2p + 1) + p \\ &= \frac{mk^2 + m + k - 1}{2}. \end{aligned}$$

Also, for even values of k, we have

$$\begin{aligned}
rc_k(P^m) &\geq (|D_k| - 2p - 1) + (\alpha + \beta) + 2\sum_{i=0}^{p} |L_i|(p-i) \\
&\geq (2pm + 1 - 2p - 1) + p + 2\sum_{i=0}^{p} |L_i|(p-i) \\
&= 2pm - p + 2p + 4m\sum_{i=1}^{p}(p-i) \\
&= 2pm + p + 4m\left(\frac{p(p-1)}{2}\right) \\
&= 2mp^2 + p \\
&= \frac{mk^2 + k}{2}.
\end{aligned}$$

This completes the proof of the lower bound. □

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