Basics of Graph Homomorphisms

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Digraph: A digraph G is a finite set V = V (G) of vertices, together with a binary relation E = E(G) on V. The elements (u, v) of E are called the arcs of G.

Symmetric digraph:

Reflexive digraph:

Oriented graph: A digraph is an oriented graph if and only if it has no symmetric pair of arcs

Underlying graph: The graph with the same vertices as G, in which{u, v} is an edge whenever at least one of (u, v), (v, u) is an arc of G

Homomorphism: A homomorphism of G to H, written as $f : G \longrightarrow H$ is a mapping $f : V(G) \longrightarrow V(H)$ such that $f(u)f(v) \in E(H)$ whenever $uv \in E(G)$.

Observation: A homomorphism of digraphs $G \rightarrow H$ is also a homomorphism of the underlying graphs, but not conversely

Walk: A walk in a graph G is a sequence of vertices v_0, $v_1, \cdots,$ v_k of G such that v_{i-1} and v_i are adjacent, for each i =1, 2,…., k.

Path: A path in G is a walk in which all the vertices are distinct

Proposition: A mapping $f : V(P_k) \longrightarrow V(G)$ is a homomorphism of P_k to G if and only if the sequence $f(0), f(1), \cdots, f(k)$ is a walk in G

Distance: $d_G(u,v)$ is the distance (length of a shortest path) from u to v in G

Proposition: If $f : G \to H$ is a homomorphism, then $d_H(f(u), f(v)) \leq d_G(u,v)$, for any two vertices u, v of G.

Cycle: A cycle in a graph G is a sequence of distinct vertices $v_1, v_2, \cdots, v_k$ of G such that each v_i, $i=2, 3, \cdots, k$, is adjacent to v_{i-1}, and v_1 is adjacent to v_k.

Proposition: A mapping $f : V(C_k) \to V(G)$ is a homomorphism of C_k to G if and only if $f(0), f(1), \cdots, f(k-1)$ is a closed walk in G.

Corollary: $C_{2k+1} \rightarrow C_{2l+1}$ if and only if $l \leq k$.

Proposition: Let G and H be digraphs, and $f : G \longrightarrow H$ a homomorphism. If $v_0, v_1, \cdots, v_k$ is a walk in G, then $f(v_0), f(v_1), \cdots, f(v_k)$ is a walk in H, of the same net length.

Homomorphisms generalize colorings

Proposition: Homomorphisms $f : G \rightarrow K_k$ are precisely the k-colorings of G.

Corollary: If $G \rightarrow H$, then $\chi(G) \leq \chi(H)$.

Theorem: For any positive integers k, l there exists a graph of chromatic number k, and girth at least l.

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Retraction: Suppose that the digraph H is a subgraph of the digraph G. A retraction of G to H is a homomorphism $r : G \to H$ such that $r(x) = x$ for all $x \in V(H)$.

Core: A core is a digraph which does not retract to a proper subgraph

Proposition: A digraph is a core if and only if it is not homomorphic to a proper subgraph.

Observation: Core of a graph G is unique up to isomorphism.

Thank You