Introduction to Category Theory

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References:

Chapter 1

① Basic Algebra Vol 2 by Jacobson

② An Introduction to Homological Algebra by J. Rotman

Motivation:

"unify" mathematical objects,

— Graphs

— Topology

— Algebra

~ 1945 by Eilenberg, MacLane.

— Cartan.

§1: Categories

Def^n: A category $\mathcal{C}$ consists of

$(ob\,\mathcal{C}, \mathrm{Hom}, \circ)$

1. A class of $Ob\ \mathcal{C}$ of objects (usually denoted as A, B, C etc).
2. For each ordered pair of objects A, B, a set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ where elements are called morphisms with domain A & codomain B.

3. $\circ$: $\mathrm{Hom}_{\mathcal{C}}(A,B) \times \mathrm{Hom}_{\mathcal{C}}(B,C) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A,C)$

$$(f, g) \longmapsto g \circ f$$

$$A \xrightarrow{f} B \xrightarrow{g} C \quad \rightsquigarrow \quad A \xrightarrow{g \circ f} C$$

s.t.

C1: If $(A,B) \neq (C,D)$ then $\mathrm{Hom}_{\mathcal{C}}(A,B) \neq \mathrm{Hom}_{\mathcal{C}}(C,D)$.

C2: (Associativity):

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

$\forall$ $f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$
$g \in \mathrm{Hom}_{\mathcal{C}}(B,C)$
$h \in \mathrm{Hom}_{\mathcal{C}}(C,D)$

C3: (Unit) $\exists$ $1_A \in \mathrm{Hom}_{\mathcal{C}}(A,A)$

s.t.

$$A \xrightarrow{1_A} A \xrightarrow{f} B \qquad f \circ 1_A = f$$

$\forall f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$

$$A \xrightarrow{f} B \xrightarrow{1_B} B \qquad 1_B \circ f = f.$$

Example: ① **Set**:

$(\text{Sets}, \text{Hom}, \circ)$

$\text{Ob}\ \mathcal{G} = \text{sets}$

$\forall\ A, B \in \text{Ob}\ \mathcal{G}$

$\text{Hom}_{\mathcal{G}}(A, B) =$ the set of function from A to B

$\circ$: — usual composition of functions.

Here $1_A \in \text{Hom}_{\mathcal{G}}(A, A)$

$$1_A : A \longrightarrow A$$
$$x \longmapsto x$$

② **Groups** $= \mathcal{G}$

$\text{Ob}\ \mathcal{G} =$ groups.

$G, H \in \text{Ob}\ \mathcal{G}$

$\text{Hom}_{\mathcal{G}}(G, H) =$ set of all group homomorphisms from G to H.

$\circ$ — composition of functions

Example: "dual category"

Let $\mathcal{C}$ be a category.

We define "category" $\mathcal{C}^{op}$ as follows.

$$\operatorname{Ob} \mathcal{C}^{op} = \operatorname{Ob} \mathcal{C}$$

For $A, B \in \operatorname{Ob} \mathcal{C}^{op}$

$$\operatorname{Hom}_{\mathcal{C}^{op}}(A, B) := \operatorname{Hom}_{\mathcal{C}}(B, A)$$

$$f \in \operatorname{Hom}_{\mathcal{C}^{op}}(A, B)$$

$$g \in \operatorname{Hom}_{\mathcal{C}^{op}}(B, C)$$

$$f \circ_{\mathcal{C}^{op}} g := f \circ_{\mathcal{C}} g$$

in $\mathcal{C}^{op}$

$$C \xrightarrow{g} B \xrightarrow{f} A$$

Here $\mathcal{C}^{op}$ is a category.

§ Functors:

Def^n: Let $\mathcal{C}, \mathcal{D}$ be categories

A functor (covariant) $F : \mathcal{C} \longrightarrow \mathcal{D}$

consists of

1. $\forall A \in Ob(\mathcal{C})$

$\exists \; FA \in Ob \; \mathcal{D}$

2. $f \in Hom_{\mathcal{C}}(A, B)$ $\quad Hom_{\mathcal{C}}(A,B) \longrightarrow Hom_{\mathcal{D}}(FA, FB)$

$f \longmapsto Ff$

then

$\exists \; Ff \in Hom_{\mathcal{C}}(FA, FB)$

$$A \xrightarrow{f} B \quad \rightsquigarrow \quad FA \xrightarrow{Ff} FB$$

s.t.

F1: $A \xrightarrow{f} B \xrightarrow{g} C \qquad FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC$

$$F(g \circ_{\mathcal{C}} f) = Fg \circ_{\mathcal{D}} Ff$$

F2: $A \xrightarrow{1_A} A \qquad FA \xrightarrow{1_{FA}} FA$

$$F(1_A) = 1_{FA}$$

Contravariant functor:

is a covariant functor from $\mathcal{C}^{op}$ to $\mathcal{D}$.

More precisely,

1. $\forall A \in \mathcal{C}$, $\exists FA \in \mathcal{D}$
2. If $A \xrightarrow{f} B$, then $\exists Ff$,

$$FB \xrightarrow{Ff} FA$$

s.t

F1: $F(g \circ f) = F(f) \circ Fg$

F2: $F(1_A) = 1_{FA}$.

Example: $\{\mathcal{C} = \text{Vector space}/\mathbb{R}\}$ $\{\text{Vector space}\}$

$$F(V) = V^*$$

$$V \xrightarrow{T} W \rightsquigarrow W^* \xrightarrow{FT} V^*$$

Example: ① $\mathcal{C}$ = Category of groups
$\mathcal{D}$ = " " sets

$$F : \mathcal{C} \longrightarrow \mathcal{D}$$

$$A \in Ob(\mathcal{C})$$

$FA := A$

$A, B \in ob(\mathcal{C})$

$\cancel{A} \xrightarrow[f]{} B$

$$\begin{array}{ccc} A & & B \\ \| & & \| \\ Ff : FA & \xrightarrow[Ff]{} & FB \end{array}$$

Define $Ff = f$

$\therefore$ F is a covariant functor.

"forgetful" functor

(2) F: Rings $\longrightarrow$ Groups

$F(R, +, \cdot) = (R, +)$

$Ff = f$

(where $f : R \longrightarrow S$ ring homomorphism)

(3) $\mathcal{C}$ = Category of topological spaces

$\mathcal{D}$ = " " groups

$\pi_1 : \mathcal{C} \longrightarrow \mathcal{D}$

$X \longmapsto \pi_1(X)$

fundamental group of X.

$$X \xrightarrow{f} Y \qquad \pi_1(X) \xrightarrow{\pi_1(f)} \pi_1(Y)$$

§3. Products

Let G_1, G_2 - groups.

product of objects - " product of G_1, G_2"

$G_1 \times G_2$

Defⁿ: Let $\mathcal{C}$ be a category

Given $A_1, A_2 \in Ob(\mathcal{C})$.

We define a product of A_1, A_2 is a triplet (A, p_1, p_2), where $A \in Ob(\mathcal{C})$

$A \xrightarrow{p_1} A_1$, $A \xrightarrow{p_2} A_2$, s.t.

Given (B, f_1, f_2), $B \in Ob(\mathcal{C})$

$f_i \in Hom_{\mathcal{C}}(B, A_i)$

$\exists !\ f \in Hom_{\mathcal{C}}(B, A)$

$$\begin{tikzcd} B \arrow[r, dashed, "f"] \arrow[d, "f_i"'] & A \arrow[dl, "p_i"] \\ A_i \arrow[ur, phantom, "\circlearrowleft"] & \end{tikzcd}$$

i.e. $p_i \circ f = f_i$

Example: ① $\mathcal{C}$ - Set

product of $A_1, A_2 = (\underbrace{A_1 \times A_2}_{A}, p_1, p_2)$

$A_1 \times A_2 := \{(a_1, a_2) : a_i \in A_i\}$

$$p_1 : A \longrightarrow A_1$$
$$(a_1, a_2) \longmapsto a_1$$

$$p_2 : A \longrightarrow A_2$$
$$(a_1, a_2) \longmapsto a_2$$

(B, f_1, f_2)

$$\begin{tikzcd}
B \arrow[r, dashed, "f"] \arrow[d, "f_i"'] & A \arrow[dl, "p_i"] \\
A_i &
\end{tikzcd}$$

Define
$$f(b) = (f_1(b), f_2(b)) \in A$$

(2) $\mathcal{C}$ = Groups.
$G, H \in Ob\ \mathcal{C}$

product of G, H:
$$= (G \times H, p_1, p_2)$$

__Facts:__ If product of A_1, A_2 exists in a category $\mathcal{C}$, then it is unique upto unique isomorphism.

"one can define product of A_α, $\alpha \in I$, $A_\alpha \in Ob\ \mathcal{C}$" where I is any indexing set.

§ Coproducts

Let $A_1, A_2 \in Ob(\mathscr{C})$.

Then a coproduct of A_1, A_2 is

a triplet (A, i_1, i_2), $A \in Ob(\mathscr{C})$

$i_i : A_i \longrightarrow A$ s.t.

given (B, g_1, g_2), $B \in Ob(\mathscr{C})$,

$g_i : A_i \longrightarrow B$, then $\exists !\, g$ s.t.

$$\begin{tikzcd} A \arrow[r, dashed, "\exists g"] & B \\ A_i \arrow[u, "i_i"] \arrow[ur, "g_i"'] \arrow[u, phantom, shift left=4ex, "\supseteq"] & \end{tikzcd}$$

s.t. $i_i \circ g = g_i$, $i = 1, 2$.

Example: ① $\mathscr{C}$ - set

$A_1, A_2 \in \mathscr{C}$

coproduct (A, i_1, i_2)

$A := A_1 \sqcup A_2$ - disjoint union

$i_i \quad A_i \hookrightarrow A$

$x \longmapsto x$

(B, g_1, g_2)

$$\begin{tikzcd} A \arrow[r, dashed, "g"] & B \\ A_i \arrow[u, "i_i"] \arrow[ur, "g_i"'] & \end{tikzcd}$$

$$g(a) = \begin{cases} g_1(a), & a \in A_1 \\ g_2(a), & a \in A_2 \end{cases}$$

$\therefore$ (A, i_1, i_2) coproduct of A_1 ; A_2.

2 $\mathcal{G}$ - Groups.

G, H groups.

coproduct - $(G \oplus H, i_1, i_2)$

$G \xrightarrow{i_1} G \oplus H$ $\qquad$ $i_2 : H \to G \oplus H$

$g \longmapsto (g, 1_H)$ $\qquad$ $h \longmapsto (1_G, h)$