Tutorial 1

1. Show that C_{2m} admits a homomorphism to C_{2n} for all $m, n \geq 2$.

2. Let $G \to H$. Then

   (i) Show that $\chi(G) \leq \chi(H)$.

   (ii) Show that $\chi(G) = \min\{|V(G)| : G \to H\}$.

   (iii) Show that $og(G) \geq og(H)$, where $og(X)$ is the length of a smallest odd cycle of X.

3. Let G be a bipartite graph. Show that $G \to K_2$.

4. Let the class of all graphs be objects and let a morphism of G to H be a homomorphism. Show that this defines a category.

5. Let the class of all simple graphs be objects and let homomorphism of G to H play the role of a categorical morphism. Show that this defines a category.

6. Let the class of all directed graphs be objects and let homomorphism of G to H play the role of a categorical morphism. Show that this defines a category.

7. Let the class of all oriented graphs (*directed graphs without loops or opposite arcs*) be objects and let homomorphism of G to H play the role of a categorical morphism. Show that this defines a category.

8. Let G, H be graphs. Let $G \leq H$ if $G \to H$. Moreover, if $G \leq H$ and $H \leq G$, then $G = H$. Also, if $G \leq H$ and $G \neq H$, then we denote it by $G < H$. This is called the homomorphism order.

   (i) Show that $<$ defines a partial order on graphs.

   (ii) Give example of a $G = H$ where G and H are not isomorphic.

   (iii) Can you find infinitely many non-isomorphic graphs which are equal (with respect to the homomorphism order)?

   (iv) Show that there exists no graph X satisfying $K_1 < X < K_2$.

9. Let G, H, K be graphs. Then

   (i) Show that $G \times H \cong H \times G$.

   (ii) Show that $G \times (H \times K) \cong (G \times H) \times K$.

   (iii) Show that $G \times (H + K) \cong (G \times H) + (G \times K)$.

10. Given two graphs G and H, their *exponential graph* H^G is a graph whose vertices are functions from $V(G) \to V(H)$ and two vertices (functions) f_1, f_2 are adjacent if $f_1(x)f_1(y) \in E(H)$ for all $xy \in E(G)$. Then

(i) Show that $f : V(G) \to V(H)$ is a homomorphism if and only if f has a loop in H^G.

(ii) Show that $H^{G+K} \cong H^G \times H^K$.

(iii) Show that $H^{G \times K} \cong (H^G)^K$.

(iv) Show that $G \times K \to H$ if and only if $F \to H^G$.

11. Show that categorical product exists for the category of directed and oriented graphs. Moreover, provide a construction to describe the product.

12. Let G and H be digraphs, and $f : V(G) \to V(H)$ is a homomorphism. If $v_0, v_1, v_2, \cdots, v_k$ is a walk in g, then $f(v_0), f(v_1), f(v_2), \cdots, f(v_k)$ is a walk in H, of the same net length.

13. A digraph with n vertices is acyclic if and only if $G \to \overrightarrow{T}_n$, where the transitive tournament $\overrightarrow{T}_n$ has vertices $1, 2, 3, \cdots, k$ and arcs ij for all $i < j$.

14. A digraph is a core if and only if it is homomorphic to a proper subgraph.

    [Note: Suppose that the digraph H is a subgraph of the digraph G. A **retraction** of G to H is a homomorphism $r : G \to H$ such that $r(x) = x$ for all $x \in V(H)$. A **core** is a digraph which does not retract to a proper subgraph.]