Basics of Binary Relations & Orders

Graph Homomorphism Lecture Series

17-AUG-2021

Binary Relations: Let A and B be two sets. A binary relation R between A and B is a subset of $A \times B$.

Here $A \times B$ is set of ordered pairs defined as follows:

$$A \times B = \{(a, b) \mid a \in A,\ b \in B\}.$$

Thus $R \subseteq A \times B$.

Ex: Let D be the set of districts and S be the set of states.

Then $R = \{(d,s) \mid d \in D, s \in S, d$ is a district in the state $s\}$.

bin. rel

Observe that $R \subseteq D \times S$.

In the defn. of bin. rel R if we have $A = B$, we say R is a relation on A.

Note that we drop the term "binary".

Properties of Relations:

Reflexive: A relation R on a set A is reflexive if $(a,a) \in R$ $\forall a \in S$.

$\forall a \in A$, $(a,a) \in R \Rightarrow R$ is reflexive.

Symmetric: A relation R on a set A is symmetric if $(b,a) \in R$ whenever $(a,b) \in R$, for $a,b \in A$.

R is symmetric if

$\forall a \forall b \left((a,b) \in R \Rightarrow (b,a) \in R\right)$.

Antisymmetric: A relation R on set A is anti-symmetric if for all $a, b \in A$, whenever $(a,b) \in R$ and $(b,a) \in R$, then $a = b$.

R is antisymmetric if
$\forall a \forall b \left((a,b) \in R \wedge (b,a) \in R \Rightarrow a = b \right)$.

Note that this defn.. says that (a,b) and (b,a) both cannot be in R when a, b are two distinct elements of A.

Transitive: A relation R is transitive if whenever $(a,b) \in R$, $(b,c) \in R$ then $(a,c) \in R$ $\forall a, b, c$.

A relation R is transitive if
$\forall a, b, c \ ((a,b) \in R \wedge (b,c) \in R \Rightarrow (a,c) \in R)$.

Examples: "divisibility" on the set of positive integers.

divisibility $(|)$, $a|b$: a divides b.

$|$: is reflexive $a|a$, symmetric ✗, transitive.

antisymmetric

Here every relation will be defined on $\{1,2,3,4\}$.

Ex:
$R_1 = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$. ← not reflexive
(3,3) is absent

not symmetric
(4,3) is absent

not transitive
(3,4), (4,1) is present & (3,1) is absent.

$R_2 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$.

This is transitive.

$\underset{R_2}{(3,2)}, (2,1) \in R_2$ and $(3,1) \in R_2$

$\underset{R_2}{(4,2)}, \underset{R_2}{(2,1)}$ and $(4,1) \in R_2$

$\underset{R_2}{(4,3)}, \underset{R_2}{(3,1)}$ and $\underset{R_2}{(4,1)}$

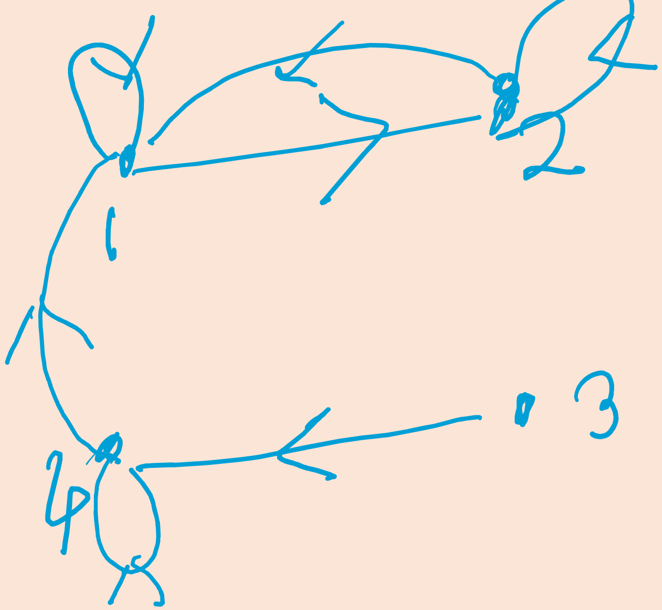
Representations using Digraphs:

R is a relation on set A.

We define a digraph $G = (V, F)$ representing R where $V = A$, F is the set of directed edges in G s.t. $(a,b) \in F$ iff $(a,b) \in R$

We say that $(a,b) \in F$ whenever there is a directed edge from a to b.

$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$



- A relation is reflexive if each vertex has a self-loop
- A relation is symmetric iff every edge that is present in the graph is bidirected.

- The relation is antisymmetric iff there are no bidirected edges.
- Whenever 3 vertices take part in a directed triangle, the triangle should be acyclic.

Counting relations:

reflexive: A reflexive relation contains $(a,a), \forall a \in A$ and it can contain any other pairs.

Total # pairs $= n^2$; # pairs that has to be present is n.

Hence # reflexive $= 2^{n^2-n}$.

symmetric : Whenever there is an edge in the graph it should be bidirected $\Rightarrow$ Number of different directed graphs on a fixed set of labeled vertices $\Rightarrow$ $2^{\frac{n^2-n}{2}} \times \underbrace{2^{n}}$ $\leftarrow$ # choice of reflexive pairs.

$\#$ antisymmetric relations:

(Whenever there is an edge between two distinct vertices we should have it in exactly one direction) I, and (for the loops we can have or not have) II $\Rightarrow 2^n$ | $3^{\frac{n(n-1)}{2}} \times 2^n$.

I

$3^{\frac{n(n-1)}{2}}$

Equivalence Relations: A relation on a set A is an equivalence relation if it is reflexive, symmetric and transitive.

Ex: $R = \{(a,b) \mid a \equiv b \pmod{m}\}$, m is an integer.

Reflexive: $a-a=0$ is divisible by m.

Symmetric: $(a,b) \in R \Rightarrow a-b = km$ for some integer k.

$\Downarrow$ $b-a = (-k)m \Rightarrow (b,a) \in R$.

transitive:

$(a,b) \in R$, $(b,c) \in R \Rightarrow a-b = k_1 m$, $b-c = k_2$, k_1, k_2 are integers

$a-b+b-c = a-c = (k_1+k_2)m \Rightarrow (a-c) \in R$.

Ex: "divides" relation, "<" less than are not equivalence relations because they are not symmetric.

(Non)

Ex: (Non) R on set of real numbers:

xRy iff $|x-y| < 1$. R is not an equiv relation

R is not transitive.

<u>Defn</u>: An equivalence class of an equiv relation R on S is a maximal subset T of S s.t. all pairs in T are related by R.

Claim: $(x,y) \in R$ iff x,y belong to the same equiv class.

Pf sketch:

- $x,y \in$ same equiv class $\Rightarrow (x,y) \in R$ from defn of equiv class.

Assume for contradiction,

- $(x,y_1) \in R$, $(x,y_2) \in R$, but $y_1 \in T_1 \subsetneq S$
  $x \in T_1 \subsetneq S$
  $y_2 \in T_2 \subsetneq S$.

T_2 does not contain y_1.

We will complete the proof once we establish that T_1 is not a maximal subset of S, where all the pairs are related. This is established by proving that every element of T_1 is related to y_2.

$(x, y_2) \in R \Rightarrow (y_2, x) \in R$ since R is symmetric.

$(y_2, x) \in R$, $(x, z) \in R \Rightarrow (y_2, z) \in R$, since R is transitive, where z is some element in T_1. Thus we can add y_2 to T_1. Hence T_1 is not maximal

The above claim implies equiv. classes partitions S.
A partition of S corresponds to an equiv class.
Here the equiv. relation is: $(x,y) \in R$ iff
$x,y \in$ same partition of
S.

Partial Ordering: A relation R on a set S is called a partial order if it is reflexive, antisymmetric and transitive.

A partially ordered set or poset is denoted by

(S, R)

ground set

Ex: divisibility relation ("|") on the set of positive integers.

Why positive? $-7|7$, $7|-7$, $7 \neq -7$

Ex: "$\leq$" on the set of integers.

$a \leq b$, $b \leq a \Rightarrow a = b$ ← anti-symmetric

Ex: set inclusion "$\subseteq$", is a partial ordering on the powerset of S. $A \subseteq B$, $B \subseteq A \Rightarrow A = B$ ← anti-symmetric.

Poset are represented using <u>Hasse diagrams</u>.

The elements a, b of a poset $(S, \leq)$ are
relation

<u>Comparable</u> iff either $a \leq b$ or $b \leq a$ and

o.w. they are <u>not comparable</u>.

When every two elements in a poset are comparable we call it a linear order.
(total)

Dense Order:

A partial or a total order on X is dense if $\forall x$ and y in X for which $x < y$, $\exists z \in X$ s.t. $x < z < y$.

e.g. rational numbers, real number with $<$ (less than) ordering. (total order).

$(x, y) \leq (x', y')$ iff $x \leq x'$ and $y \leq y'$ (partial order not total order).

Order isomorphism:

Given two posets $(S, \leq_S)$, $(T, \leq_T)$ an order isomorphism from $(S, \leq_S)$ to $(T, \leq_T)$ is a bijection f from S to T with the following property:

$\forall x, y \in S$, $x \leq_S y$ iff $f(x) \leq_T f(y)$.

Ex:

Negation is an order isomorphism for $(\mathbb{R}, \leq)$ to $(\mathbb{R}, \geq)$.

Non-ex: $((0,1), \leq)$, $([0,1], \leq)$

less than less than

There is no order isomorphism between the above two sets.

There is no least element of $(0,1)$ but there is a least element in $[0,1]$.

Suppose $f(x) \to 0$ for some $x \in (0,1)$.

$$y \leq x \leftrightarrow \underbrace{\underset{[0,1]}{\underset{\cap}{f(y)}} \leq \underset{0}{\underset{\shortparallel}{f(x)}}}_{\text{This is impossible.}}$$