Graph homomorphisms

some results

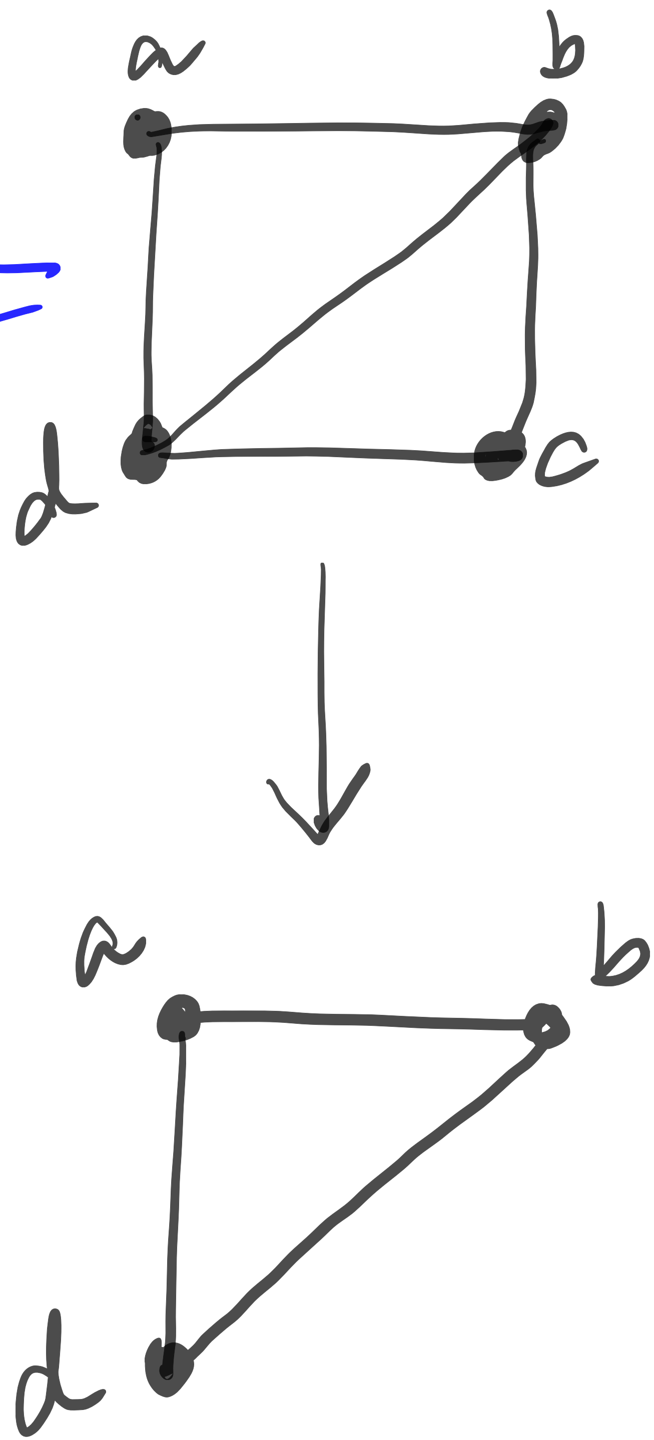
Core: Soumen will give.

$\Updownarrow$

Core: G is a core if G does not admit a homomorphism to any of its proper subgraph.

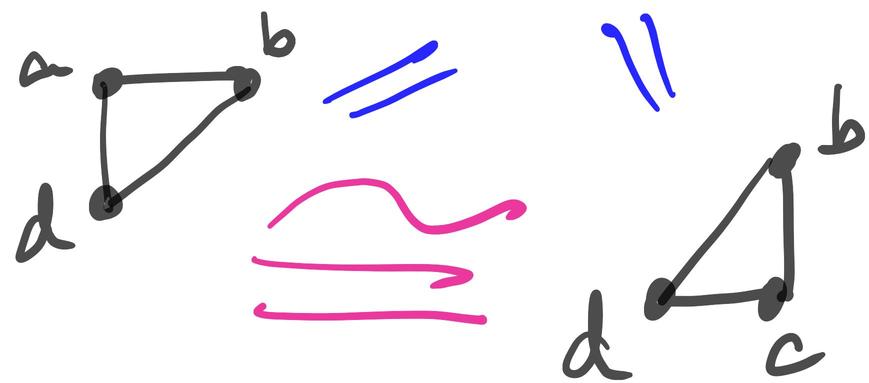
$\text{core}(G) =$ minimal subgraph H of G s.t. $G \longrightarrow H$.

$G =$



is this a core?

What is core(G)?



Thm: core(G) is unique upto isomorphism.

Pf: Suppose A & B are both cores of G.

$$
\begin{tikzcd}
B \arrow[r, hook, "\text{inj}"'] \arrow[rr, bend left, "\psi"] & G \arrow[r, two heads, "\text{surj}"'] & A
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \arrow[r, hook, "\text{inj}"'] \arrow[rr, bend right, "\phi"'] & G \arrow[r, two heads, "\text{surj}"'] & B
\end{tikzcd}
$$

$$
\begin{tikzcd}
U, V \arrow[r, mapsto] & Z \arrow[r, mapsto] & W
\end{tikzcd}
$$

Assume

$$
\begin{tikzcd}
A \arrow[r, "\phi"', "\text{not inj}"] \arrow[rr, bend right, "\alpha", "\text{not inj}"'] & B \arrow[r, "\psi"] & A
\end{tikzcd}
$$

$$\Rightarrow \quad A \xrightarrow{\alpha} A \text{ is not surjective.}$$

$$
\begin{tikzcd}
G \arrow[d] \arrow[rr, bend left] & & \\
A \arrow[rr] & & A' \subsetneq A
\end{tikzcd}
\Rightarrow \Leftarrow \; A \text{ is core}
$$

Therefore $A \xrightarrow[\text{inj}]{} B \Rightarrow A \subseteq B$

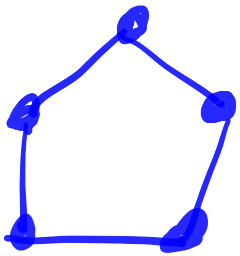
<u>Illy</u>

$$B \xrightarrow[\text{inj}]{} A \Rightarrow B \leq A.$$

$$\Rightarrow A \simeq B$$

$\square$

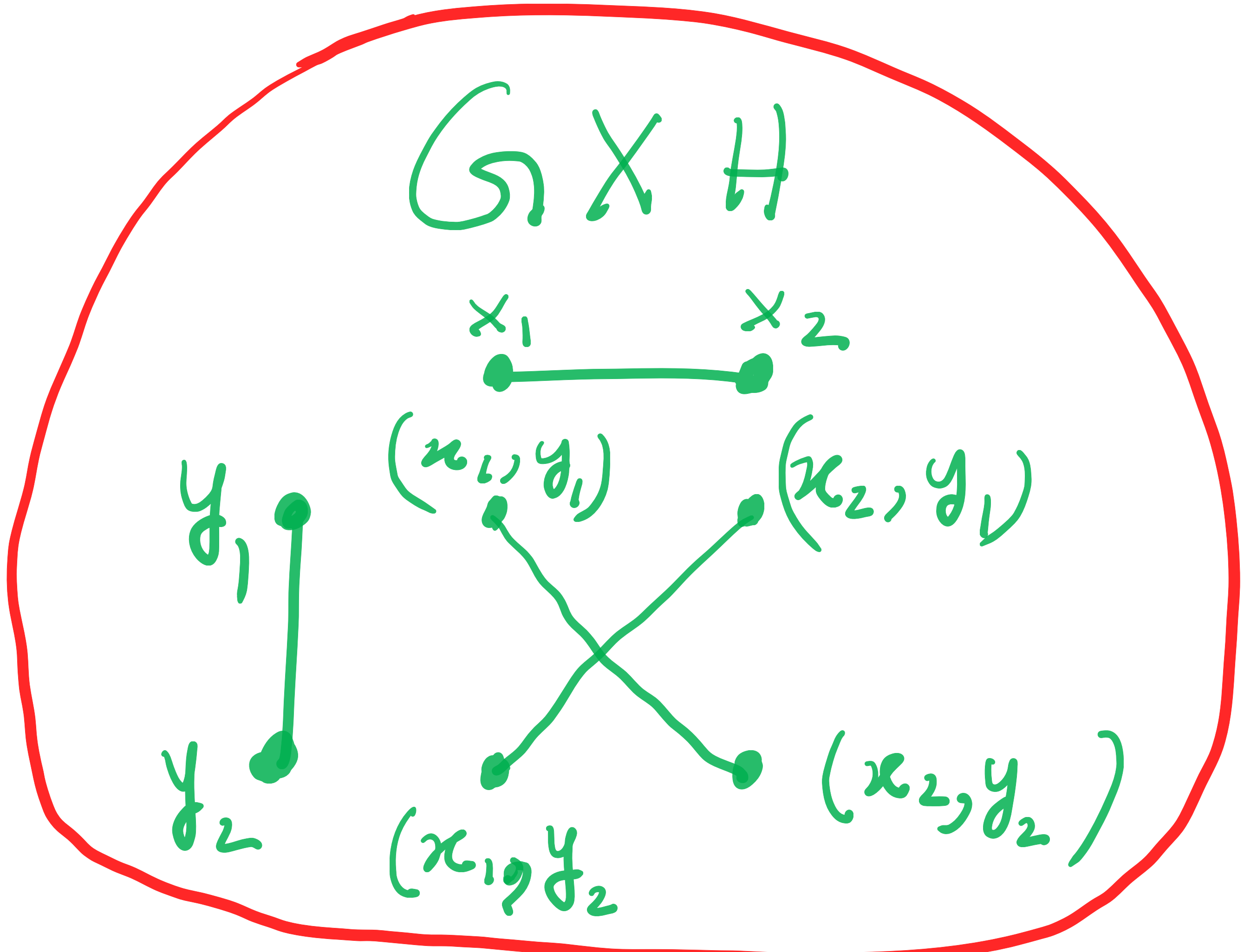
Example: Core that is not K_n

 , C_{2n+1}

$G \longrightarrow H$

$\Rightarrow \chi(G) \leq \chi(H)$

Th 2: The categorical product exists.



Th^{3}: Let $G \neq K_1$ or $H \neq K_2$.

Then $\exists\, X$ s.t. $G < X < H$

Pf: Later.

$$\boxed{\left.\begin{array}{l} {}^{*}\,A \to B \\ B \not\to A \end{array}\right\} := A < B}$$

Thm (Lovász): $G \cong H$ iff

$$|\mathrm{Hom}(X,G)| = |\mathrm{Hom}(X,H)| \quad \forall X.$$

$^*\mathrm{Hom}(A,B)$ = set of all homomorphisms of $A \to B$

Let us enumerate all graphs as $X_1, X_2, \ldots$

Let the Lovász vector of G be $(g_1, g_2, g_3, \ldots)$, $g_i = |\mathrm{Hom}(X_i, G)|$.

Thm (Lovász): $G \cong H$ iff their Lovász vectors are the same.

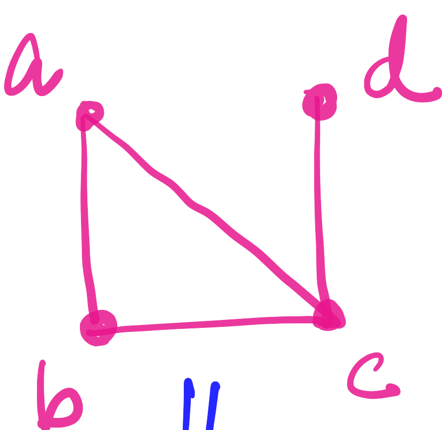
Conj (edge reconstruction)

$G \cong H$ iff $\mathrm{Deck}(G) = \mathrm{Deck}(H)$

* $\mathrm{Deck}(A) = \{A \setminus e \mid e \in E(A)\}$

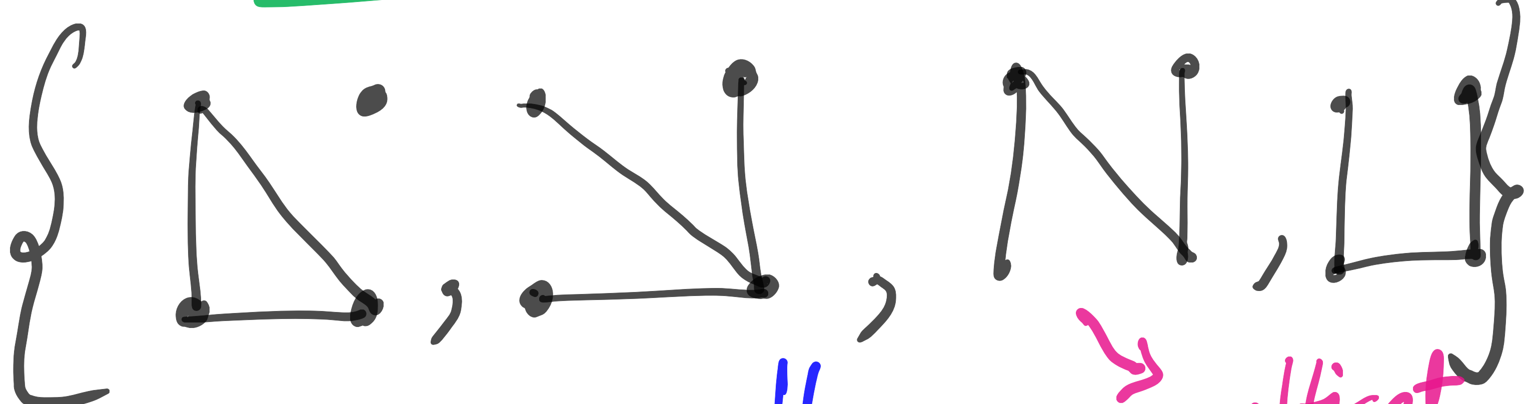
↳ multiset.

Example!



$\parallel$

A



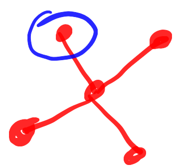
$\parallel$

$\mathrm{Deck}(A)$

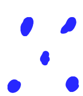
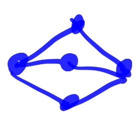
multiset

Conj (switching reconstruction / Stanley's)

$G \cong H$ iff $\mathrm{swDeck}(G) = \mathrm{swDeck}(H)$ for $|V(G)| \neq 4$.

Example: $A =$ 

$\mathrm{swDeck}(A) =$

$\{$  , 4  $\}$

↳ multiset.

Known: True for all

$|V(G)| \not\equiv 0 \pmod 4$

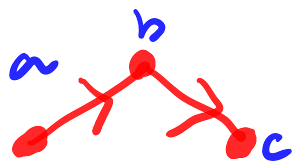
* switching a vertex v of a graph: convert edges $\longleftrightarrow$ non-edges for the ones incident to v.

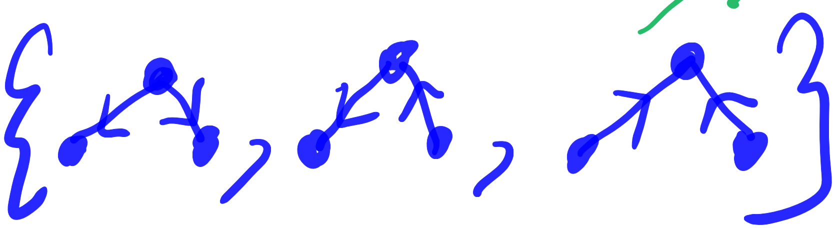
$\mathrm{swDeck}(G) = \{ G \text{ with } v \text{ switched} \mid v \in V(G) \}$

multiset.

Conj (switching reconstruction digraphs)

$\vec{G} \cong \vec{H}$ iff $\mathrm{swDeck}(\vec{G}) = \mathrm{swDeck}(\vec{H})$ for $|V(G)| \neq 4, 8$.

Example:  $= A$

$\mathrm{swDeck}(A) =$ 

multiset

Known: True for all $|V(G)| \not\equiv 0 \pmod 4$

* switching a vertex v of a digraph: convert in-arcs $\longleftrightarrow$ out-arcs for the ones incident to v.

$\mathrm{swDeck}(G) = \{ \vec{G} \text{ with } v \text{ switched} \mid v \in V(\vec{G}) \}$

multiset.

Thanks