Proving the conjecture on (n,m)-absolute clique number of planar graphs

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Abstract

An (n,m)-graph G is a graph that has n types of arcs and m types of edges. The (n,m)-chromatic number of an (n,m)-graph G is the smallest order of an (n,m)-graph H such that there exists a homomorphism that is a type (and direction) preserving vertex-mapping of G to H. An (n,m)-absolute clique C is an (n,m)-graph such that its (n,m)-chromatic number of C is its order itself. Bensmail, Duffy and Sen [Graphs and Combinatorics 2017] conjectured that if C is a planar absolute (n,m)-clique then it has at most $3(2n+m)^2+(2n+m)+1$ vertices for all $(n,m) \neq (0,1)$. In this paper, we positively settle the conjecture for all $(n,m) \neq (0,1), (1,0)$ and $(0,2)$. This, along with the existing proofs for $(n,m) = (1,0)$ and $(0,2)$ due to Nandy, Sopena and Sen [Journal of Graph theory 2016] completes the proof of the conjecture for all values of $(n,m) \neq (0,1)$.

Keywords: colored mixed graphs, planar graphs, homomorphisms, chromatic number, absolute clique number.

1 Introduction and the main results

The concept of (n,m)-graphs and their homomorphisms were introduced by Nešetřil and Raspaud [10] as a generalization of the notion of m-edge colored graphs [1] and oriented graphs [14]. An (n,m)-graph is a graph having n different types of arcs and m different types of edges. We denote the set of vertices, arcs and edges of an (n,m)-graph G by $V(G)$, $A(G)$, and $E(G)$, respectively. In the context of (n,m)-graphs, $(0,1)$-graph is an undirected graph, a $(1,0)$-graph is a directed graph, and a $(0,m)$-graph is an m-edge-colored graph. We denote the underlying graph of an (n,m)-graph, by $und(G)$. In this article, we focus only on those (n,m)-graphs G whose $und(G)$ is a simple graph, unless otherwise stated. We follow West [15] for standard graph-theoretic notations and terminology.

Given two (n,m)-graphs G and H, a vertex mapping $f : V(G) \to V(H)$ is a *homomorphism* of G to H if for every arc (resp., edge) uv in G, $f(u)f(v)$ is also an arc (resp., edge) in H, having the same type as uv. There are three important parameters related to the study of homomorphism of an (n,m)-graph G, namely, the (n,m)-chromatic number $\chi_{n,m}(G)$, the (n,m)-relative clique number $\omega_{r(n,m)}(G)$, and the (n,m)-absolute clique number $\omega_{a(n,m)}(G)$. The (n,m)-chromatic number $\chi_{n,m}(G)$ of G is the minimum $|V(H)|$ such that G admits a homomorphism to H, the (n,m)-relative clique number $\omega_{r(n,m)}(G)$ is the maximum cardinality of a vertex subset $R \subseteq V(G)$, called an (n,m)-*relative clique* such that $f(u) \neq f(v)$ for any homomorphism f of G to an (n,m)-graph H, and the (n,m)-absolute clique number $\omega_{a(n,m)}(G)$ is the maximum order of vertices of a subgraph A called an (n,m)-*absolute clique* satisfying an $\chi_{n,m}(A) = |V(A)|$. Observe that for $(n,m) = (0,1)$, the parameter $\chi_{n,m}$, is nothing but the chromatic number of simple graphs. Moreover the parameters $\omega_{r(n,m)}$ and $\omega_{a(n,m)}$, when restricted to the instance $(n,m) = (0,1)$, coincide with each other and are equivalent to the notion of clique number for simple graphs. Thus, $\chi_{n,m}$ is a generalization of the notion of chromatic number for (n,m)-graphs. On the other hand, the notion of clique number for (n,m)-graphs ramified into $\omega_{r(n,m)}$ and $\omega_{a(n,m)}$. For a family $\mathcal{F}$ of undirected simple graphs, we have,

$$p_{(n,m)}(\mathcal{F}) = \max\{p_{(n,m)}(G) : und(G) \in \mathcal{F}\},$$

where $p \in \{\chi_{n,m}, \omega_{r(n,m)}, w_{a(n,m)}\}$.

One immediate observation from the definitions of these parameters is that [2],

$$\omega_{a(n,m)}(G) \leq \omega_{r(n,m)}(G) \leq \chi_{n,m}(G).$$

In a quest to find an analogous version of the 4-Color Theorem and the Grötzsch Theorem, Marshall [8] and Raspaud and Sopena [13] proved $18 \leq \chi_{1,0}(\mathcal{P}_3) \leq 80$, where $\mathcal{P}_3$ denotes the family of planar graphs. Similarly, Ochem, Pinlou and Sen [12] established the bounds $20 \leq \chi_{0,2}(\mathcal{P}_3) \leq 80$. Moreover, a line of the study explored the values of $\chi_{1,0}(\mathcal{P}_g)$ and $\chi_{0,2}(\mathcal{P}_g)$ for all $g \geq 3$ establishing bounds, where $\mathcal{P}_g$ denotes the family of planar graphs having girth at least g.

Continuing this line of study, for $\chi_{n,m}(\mathcal{P}_3)$ a lower and upper bound, cubic [4] and quartic [11] in $(2n+m)$, respectively, were found. Moreover, an exact bound for (n,m)-chromatic number of sparse planar graphs with a very large girth was established in [7]. In the study of finding absolute clique number, Bensmail, Duffy and Sen [2] proved lower and upper bounds for the absolute (n,m)-clique number for the family of planar graphs,

Theorem 1.1 (Bensmail, Duffy and Sen, 2017 [2])**.** *For the family $\mathcal{P}_3$ of planar graphs,*

$$3(2n+m)^2 + (2n+m) + 1 \leq \omega_a(n,m)(\mathcal{P}_3) \leq 9(2n+m)^2 + 2(2n+m) + 2,$$

for all $(n,m) \neq (0,1)$.

They [2] conjectured that the (n,m)-absolute clique number of planar graphs in fact attains its lower bound.

Conjecture 1.2. *Let $\mathcal{P}_3$ denote the family of planar graphs. Then for all $(n,m) \neq (0,1)$ we have,*

$$\omega_{a(n,m)}(\mathcal{P}_3) = 3(2n+m)^2 + (2n+m) + 1.$$

A restricted version of this conjecture for $(n,m) = (1,0)$ was posed as a question by Klostermeyer and MacGillivray [6], and it was positively settled by Nandy, Sen and Sopena [9].

As we see, for the cases when $(n,m) = (1,0)$ and $(0,2)$, the conjecture was proved in [9]. In this work, we positively settle the conjecture [2] for all $(n,m) \neq (0,1), (1,0)$ and $(0,2)$.

Theorem 1.3. *Let $\mathcal{P}_3$ denote the family of planar graphs. Then for all $(n,m) \neq (0,1)$ we have,*

$$\omega_{a(n,m)}(\mathcal{P}_3) = 3(2n+m)^2 + (2n+m) + 1.$$

Thus with this result, the study of this parameter (n,m)-absolute clique number is complete for all $(n,m) \neq (0,1)$. We provide a consolidated list (see Table 1) of all bounds of these three parameters for the family of planar graphs with girth restrictions to place our work in context.

g	$\omega_{a(n,m)}(\mathcal{P}_g)$	$\omega_r(n,m)(\mathcal{P}_g)$	$\chi_{n,m}(\mathcal{P}_g)$
3	$\mathbf{3p^2+p+1}$	$\left[3p^2+p+1, 42p^2-11\right]$ [3]	$\left[p^3+\epsilon p^2+p+\epsilon, 5p^4\right]$ [4, 11]
4	p^2+2 [3]	$\left[p^2+2, 14p^2+1\right]$ [3]	$\left[p^2+2, 5p^4\right]$ [3, 11]
5	$\max(p+1,5)$ [3]	$\max(p+1,6)$ [3]	$\left[2p+1, 5p^4\right]$ [3, 11]
6	$p+1$ [3]	$\max(p+1,4)$ [3]	$\left[2p+1, 5p^4\right]$ [3, 11]
$g \geq 7$	$p+1$ [3]	$p+1$ [3]	$\left[2p+1, 5p^4\right]$ [3, 11]
$g \geq 8p$	$p+1$ [3]	$p+1$ [3]	$2p+1$ [7]

Table 1: This is the list of all known lower and upper bounds for $\omega_{a(n,m)}(\mathcal{P}_g)$, $\omega_r(n,m)(\mathcal{P}_g)$, $\chi_{n,m}(\mathcal{P}_g)$ where $\mathcal{P}_g$ denotes the family of planar graphs having girth at least g. Moreover, the list captures the bounds for all $(n,m) \neq (0,1)$ where $(2n+m)$ is denoted by p. Finally, the parameter ϵ takes the value 1 when m is an odd number or 0, and takes the value 2 otherwise.

2 Proof of the Theorem 1.3

We give necessary notations and terminologies wherever required in the course of proof. As the proof is lengthy with many calculations, we give a proof sketch of the main Theorem 1.3. Interested readers are encouraged to find the detailed proofs in `https://homepages.iitdh.ac.in/~sen/BNST.pdf`.

Any two vertices u, v in an (n,m)-graph G, can have at most $(2n+m)$-types of adjacencies. Let the set $A_{(n,m)} = \{1, 2, 3, \cdots n, (n+1), \cdots 2n, (2n+1), (2n+2), \cdots (2n+m)\}$ be all the possible types of adjacencies in any (n,m)-graph. If there is an arc of type i from u to v then we say that u is a $(2i-1)$-neighbor of v or equivalently v is a $2i$-neighbor of u for all $i \in \{1, 2, 3, \cdots n\}$. If there is an edge of type j between u and v, then we say that u is a $2n+j$-neighbor of v or equivalently v is a $2n+j$-neighbor of u for all $j \in \{1, 2, 3, \cdots m\}$. If v is an α-neighbor of u, we denote it by $u \sim_\alpha v$. The set of all α-neighbors of u are denoted by $N^\alpha(u)$. Two vertices u, v *agree* on a common neighbor z if $z \in N^\alpha(x) \cap N^\alpha(y)$ for some $\alpha \in A_{(n,m)}$, *disagrees* on z if otherwise. A *special* 2-*path* xzy is a 2-path in G where x and y disagrees on z. We recall a useful characterization by Bensmail, Duffy and Sen [2].

Proposition 2.1 (Bensmail, Duffy and Sen, 2017 [2]). *An (n,m)-graph is an (n,m)-clique if and only if every pair of non-adjacent vertices are joined by a special 2-path.*

The lower bound of the result is already established by Bensmail, Duffy, and Sen [2].

Theorem 2.2 (Bensmail, Duffy, and Sen, 2017 [2]). *There exists a planar (n,m)-graph P satisfying $\omega_a(n,m)(P) = 3(2n+m)^2 + (2n+m) + 1$ for all $(n,m) \neq (0,1)$.*

Moreover, the result is proved for the particular cases when $(2n+m) = 2$ [9]. So, we need to prove that $\omega_a(n,m)(\mathcal{P}) \leq 3(2n+m)^2 + (2n+m) + 1$ for all (n,m) satisfying $(2n+m) \geq 3$. To do so, we will consider an arbitrary planar absolute (n,m)-clique H and show that it has at most $3(2n+m)^2 + (2n+m) + 1$ many vertices. For the rest of the section, let us fix an arbitrary planar absolute (n,m)-clique H, where (n,m) satisfies the condition $(2n+m) \geq 3$. Moreover, without loss of generality, assume that H is triangulated. We can assume so because the absolute (n,m)-clique number of a (n,m)-graph is greater than or equal to any of its subgraphs.

Observation 2.3. *The underlying graph of any absolute (n,m)-clique has diameter at most 2.*

Recall that a planar graph having diameter one can have at most 4 vertices. Thus, we may assume that $und(H)$ has a diameter of exactly 2. It is known [5] that if a planar graph has a diameter 2, then its domination number is at most 2, but for a single exception of a graph on 11 vertices. As our relevant upper bound is greater than 11, we may assume that H has domination number at most 2. In fact, Bensmail, Duffy, and Sen [2] has shown that if H has domination number 1, then it must have at most $3(2n+m)^2 + (2n+m) + 1$ many vertices.

Proposition 2.4 (Bensmail, Duffy and Sen, 2017 [2])**.** *If a planar absolute (n, m)-clique has domination number one, then it has at most $3(2n+m)^2 + (2n+m) + 1$ vertices.*

In view of the above proposition, we may now assume that the domination number of H is exactly 2. Suppose that $D = \{x, y\}$ is a dominating set of size 2 of H. However, suppose that among all dominating sets of size 2 of H, D is the one for which the two vertices of D have maximum common neighbors. Before proceeding further, we will present some useful conventions used in the proof. Let $C = \{a_1, a_2, a_3, \cdots a_k\}$ be the set of all common neighbors of x and y. Let C_t^α be the set of all α-neighbors of t in C where $t \in \{x, y\}$. Let S_t be $N(t) \setminus C$, S_t^α be $N^\alpha(t) \setminus C$ for $t \in \{x, y\}$ and $S = S_x \cup S_y$. We fix a particular embedding of H in such a way that the vertices in C and the vertices in S_x are arranged in anti-clockwise direction in the increasing order of their indices around x. Let us call the region bounded by $\{x, a_i, y, a_{i+1}, x\}$ as R_i and let R_0 be the unbounded region. See Figure 1 for reference. Our goal is to prove

$$|V(H)| = |C| + |S| + |D| \leq 3(2n+m)^2 + (2n+m) + 1.$$

We also further assume $p = 2n + m$ for the rest of this proof. Our proof is contained in several lemmas and observations.

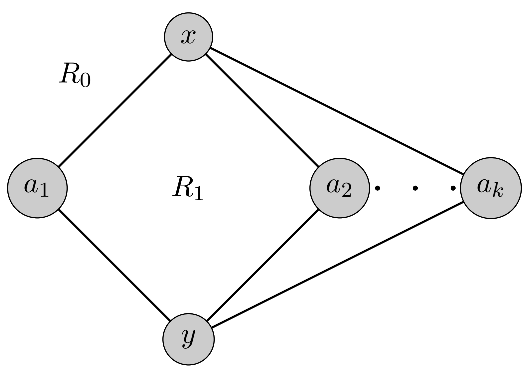

Figure 1: The planar embedding of H.

Observation 2.5. *Given any two vertices u and v of H, if either u, v are adjacent, or $|N(u) \cap N(v)| \geq 6$, then we have $|N^\alpha(u) \cap N^\beta(v)| \leq 3$ for any $\alpha, \beta \in A_{n,m}$.*

Proof. Suppose, $|N^\alpha(u) \cap N^\beta(v)| \geq 4$, and let those vertices be $a_1, a_2, \cdots a_k$, where $k \geq 4$. If u and v are adjacent, either ua_1v forms a face or ua_kv forms a face, in either case, a_1 cannot see a_4 without disturbing the planarity. Similarly, when $|N(u) \cap N(v)| \geq 6$, the only way a_1 can see a_4 is by a_6, but that will be of distance more than 2, which is not possible. Thus we get a contradiction in either case. □

Lemma 2.6. *The number of vertices in C is at most $3p^2$.*

Proof. First of all, note that, as $(2n+m) \geq 3$, the quantity $3(2n+m)^2 \geq 27$. Thus, the lemma is trivially true for $|C| \leq 5$. Fix an $\alpha \in A_{n,m}$ and consider the set C_x^α. Notice that there can be at most 3 vertices in this set with the same type of adjacency with y due to Observation 2.5. Therefore, C_x^α can have at most $3p$ vertices. Since α is arbitrary, the maximum cardinality of the set C can be of at most $3p^2$. □

Lemma 2.7. *If the number of vertices in C is at least $3p^2 - 3p + 5$, then $|S| = 0$*

Proof. Suppose there exists a $x_i \in S$ in region R_i, then x_i can see at most 4, *i.e.* $a_{i-1}, a_i, a_{i+1}, a_{i+2}$ vertices in C without the help of x. This forces x_i to see the rest of the vertices in C via x. Thus $x \sim_j x_i$, for some $j \in A_{n,m}$, implies $x \sim_t a_s$ for some $t \in A_{n,m} \setminus \{j\}$ for all $a_s \in C \setminus \{a_{i-1}, a_i, a_{i+1}, a_{i+2}\}$. Thus, C can have at most $p \cdot 3(p-1) + 4 = 3p^2 - 3p + 4$ vertices due to Observation 2.5, which is a contradiction. □

The following lemma establishes a relation between the cardinality of the set C and the set S. In particular, we prove that if C is arbitrarily small, then S is restricted to at most $3p + 1$.

Lemma 2.8. *For any $t \in \{x, y\}$, for any $\alpha \in A_{n,m}$, and $|C| = k$, we have the following:*

(i) If $|C_t^\alpha| \geq 5$ then, we have $|S_t^\alpha| = 0$.

(ii) If $|C_t^\alpha| \geq 4$ then, we have $|S_t^\alpha| \leq 2$. Moreover if $k \geq 5$, then $|S_t^\alpha| \leq 1$.

(iii) If $|C_t^\alpha| \geq i$ then, $|S_t^\alpha| \leq 3p + 1 - i$ for $k \geq 3$ and $i \in \{0, 1, 2, 3\}$. Moreover, if $|S_y| = 0$ and $|C_t^\alpha| \neq 0$, then $|C_t^\alpha \cup S_t^\alpha| \leq 3p$.

Proof. We give the proof of each cases separately as follows.

(i) Observe that a vertex $x \in S_x^\alpha$ can see at most four vertices of C without the help of x. Moreover these four vertices have to be in adjacent regions. Let $x \in S_x^\alpha$ belong to the region R_i. Then, x sees a_i, a_{i+1} directly and a_{i-1}, a_{i+2} via a_i, a_{i+1} respectively. Thus, if $|C_x^\alpha| \geq 5$, x cannot see the remaining vertices of C_x^α. Hence, we get, $|S_x^\alpha| = 0$.

(ii) Let $|C_x^\alpha| \geq 4$, then with the above proof it is clear that if $k \geq 5$, we have $|S_x^\alpha| \leq 1$. For the case when $|C_x^\alpha| = 4$ and $|C| = 4$, on the one hand, two vertices in S_x^α cannot be in the same region due to planarity. On the other hand, vertices from S_x^α cannot be present in three regions R_{i-1}, R_i, R_{i+1} as otherwise $x_i \in R_{i-1} \cap S_x^\alpha$ cannot see $x_{i+1} \in R_{i+1} \cap S_x^\alpha$ as they are at distance at least 3 from each other. Thus, we can have at most 2 vertices x_i, x_{i+1}, one in each region R_i, R_{i+1} respectively.

(iii) Let $|C_x^\alpha| = i$ for $i \in \{0, 1, 2, 3\}$, if the vertices in C_x^α are not consecutive, then the proof follows from the above two cases. Thus the interesting case is when vertices in C_x^α are consecutively placed in the planar embedding of H. We deal with the sub-cases for each $i \in \{3, 2, 1, 0\}$ separately.

   (a) Let $|C_x^\alpha| = 3$. Notice that for $k \geq 4$, S_x^α can be present only in at most two adjacent regions except for an exceptional case in which we deal separately. Let S_x^α be present in the regions be R_j, R_{j+1}. Firstly any two vertices in S_x^α cannot see each other directly, due to planarity. This forces all the vertices of $S_x^\alpha \cap R_j$ have to see each other via a_{j+1} and so do the vertices of $S_x^\alpha \cap R_{j+1}$. Suppose there are i_1-types of adjacency among the vertices in $S_x^\alpha \cap R_j$ and a_{j+1} and i_2-types of adjacencies among the vertices in $S_x^\alpha \cap R_{j+1}$ and assume $a_{j-1} \sim_{\beta_1} a_j$ and $a_j \sim_{\beta_2} a_{j+1}$ then, when $\beta_1 = \beta_2$, we get

$i_1 + i_2 \leq (p-1)$ as private neighbors of x in R_j has to see private neighbors of x in R_{j+1} via a_{j+1}. As each of the adjacency types can be present at most 3 times by Observation 2.5, we get $|S_x^\alpha| \leq 3(p-1) = 3p-3 \leq 3p-2$. Suppose when $\beta_1 \neq \beta_2$, as all the private neighbors in R_j should see the private neighbors in R_{j+1} via a_{j+1}, notice that the vertex adjacent to a_j and a_{j+1}(we call them corner vertices for convenience) may have β_1 or β_2 without any conflict. Thus, in this case, the adjacency types can be at most, $(i_1-1)+(i_2-1) \leq (p-2)$. Along with the four corner vertices, $|S_x^\alpha| \leq 3(p-2)+4 = 3p-2$. Moreover, $|S_y| = 0$, then $x \not\sim y$, as otherwise, y will be dominating vertex which is not possible. Now, say suppose, $a_{j+1} \sim_\gamma y$, and $\gamma \notin \{\beta_1, \beta_2\}$, then these i_1 and i_2 (except for the four corner vertices) cannot be of the type β_1, β_2, γ, which forces $|S_x^\alpha| \leq 3(p-3)+4 = 3p-5$. In other case, $\gamma = \beta_1$ or β_2, then, $|S_x^\alpha| \leq 3(p-2)+3 = 3p-3$. If $|S_x^\alpha|$ belong to only one region, the calculations are similar; observe that all the vertices in S_x^α have to see each other only via a_i and the adjacency types has to be different from β where $a_{i-1} \sim_\beta a_i$. Thus using Observation 2.5, we have, $|S_x^\alpha| \leq 3p-3$. In the exceptional case when S_x is in all regions, due to planarity, we can immediately see that at most one vertex in the region can be present implying $|S_x^\alpha| \leq 3$.

(b) Let $|C_x^\alpha| = 2$ and for $k \geq 3$. If $a_1 \not\sim a_3$, then we have $|S_x^\alpha| \leq 3(p-1)+2$. The two come from the corner vertices. Moreover, if $|S_y| = 0$, then, there are two cases here. In either cases, we can observe that $|S_x^\alpha| \leq 3p-2 \leq 3p$.

(c) Let $|C_x^\alpha| = 0, 1$ and for $k \geq 3$, in either cases, as $x \not\sim y$, we identify x and y, we get an outer planar graph and α-neighbors of xy form a relative clique and from [2], what we have is $|S_x^\alpha| \leq 3(p-1)+1 \leq 3p$.

□

Thus from the above lemma 2.8, we get $|C_x^\alpha \cup S_x^\alpha| \leq 3p$ if x has a private α-neighbor. Thus, if there are no private neighbors of y, then we can prove that H has at most $3p^2+p+1$ vertices.

Lemma 2.9. *If $|S_y| = 0$, then $|V(H)| \leq 3p^2+p+1$.*

Proof. If $|S_y| = 0$, then $x \not\sim y$, as otherwise, y will be a dominating vertex, which is not possible. Then triangulation of H forces the edges $a_1a_2, a_2a_3, \cdots a_{k-1}a_k$. Thus, every vertex of S has to see y via a_i or a_{i+1}. From Lemma 2.8, in all cases we have $|C_x^\alpha \cup S_x^\alpha| \leq 3p$. Therefore, $|V(H)| \leq 3p(p)+2 = 3p^2+2 \leq 3p^2+p+1$ as $p \geq 3$. □

A major part of the proof lies in showing if both x and y has private neighbors, then also $|V(H)| \leq 3p^2+p+1$. To show that, one important bound is the following.

Lemma 2.10. *If $k \geq 3$, then $|S_x^\alpha \cup S_y^\beta| \leq 3p+1$ for any $\alpha, \beta \in A_{n,m}$.*

Proof. If $k \geq 3$, if we delete the vertices x and y and look at $S_x^\alpha \cup S_y^\beta$ in every region, what we get is a outerplanar graph. Thus the set $S_x^\alpha \cup S_y^\beta \cap R_i$ induces a relative (n,m)-clique. From [2], we have the bound. □

Lemma 2.11. *For $k \geq 3$, we have $|V(H)| \leq 3p^2+p+1$.*

Proof. Suppose there are i types of adjacency present between x and vertices in S_x and j types of adjacency present between y and vertices in S_y. Suppose t many vertices in C can see each other either directly or by special 2-path or via S_x. But the rest of $k-t$ has to see each other via x,these vertices can have at most $3(p-j)$ types of adjacency with y, Putting these together should be at most p. Thus we get a bound on k,

$$i + \frac{k-t}{3(p-j)} \leq p$$

$$k \leq 3p^2 - 3pi - 3pj + 3ij + t$$

Without loss of generality, let us assume that $i \geq j$,

Suppose we have $\{\alpha_1, \alpha_2, \cdots \alpha_i\}$ types of adjacency between x and S_x, and $\{\beta_1, \beta_2, \cdots \beta_j\}$ types of adjacency between y and S_y, Since, $x \not\sim y$, As, $S_x^{\alpha_j} \cup S_y^{\beta_j}$ is an outerplanar graph. This set induces a relative (n, m)-clique. So we club α_j, β_j- private neighbors of x and y respectively and the remaining $(i-j)$-types can have at most $3p$ vertices of that corresponding types. Thus we have,

$$|S| \leq (3p+1)j + 3p(i-j) = 3pi + j$$

Using the above two equations and the fact that $i \leq p$, we get,

$$|V(H)| \leq 3pi + j + k + 2 \tag{1}$$
$$\leq 3p^2 - 3j(p-i) + j + t + 2 \tag{2}$$

Notice from (2), we are done for the case if $1 \leq j \leq i < p-1$ or if $1 < j \leq i \leq p-1$. Also when $j = 1, i = p-1$ and $t \leq 4$. Similarly, from (1), it is immediate to see if $i = p$, $j + k \leq p - 1$, we are done. Now we are left to check only when $j = 1$, $i = p - 1$ and $t \geq 5$ and the case when $i = p$ and $j + k \geq p$.

□

Lemma 2.12. *For $k = 2$, we have $|V(H)| \leq 3p^2 + p + 1$.*

Lemma 2.13. *For $k = 1$, we have $|V(H)| \leq 3p^2 + p + 1$.*

Proof of Theorem 1.3. As the graph H is triangulated and has diameter two, any dominating set $D = \{x, y\}$ must have at least one common neighbor. Therefore, using Lemmas 2.11, 2.12, and 2.13, we are done. □

Remark We are dynamically updating the proofs of the lemmas stated above; this is a preliminary version with some of the proofs.

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